ω-Regular Properties of Linear Recurrence Sequences

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$$\begin{array}{c} \mathbb{Q} \quad \mathbb{Q} \quad \mathbb{Q} \quad \mathbb{Q} \\ u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_{d-1} u_{n-(d-1)} a_d u_{n-d} \qquad n > d \end{array}$$

Linear recurrence sequences

$$u_n = \overset{\mathbb{Q}}{a_1} u_{n-1} + \overset{\mathbb{Q}}{a_2} u_{n-2} + \cdots + \overset{\mathbb{Q}}{a_{d-1}} u_{n-(d-1)} \overset{\mathbb{Q}}{a_d} u_{n-d} \qquad n > d$$

$$\underset{\mathbb{Q}^{1\times d}}{\mathbf{q}} \, \overset{\mathbb{Q}^{d\times d}}{\mathbf{M}^n} \, \underset{\mathbb{Q}^{d\times 1}}{\mathbf{p}} \qquad n \in \mathbb{N}$$

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$$u_n = \overset{\overline{\mathbb{Q}}[x]}{P_1}(n)\underset{\overline{\mathbb{Q}}}{\Lambda_1^n} + \cdots + \overset{\overline{\mathbb{Q}}[x]}{P_d}(n)\underset{\overline{\mathbb{Q}}}{\Lambda_d^n} \qquad n \in \mathbb{N}$$

For example

$$u_n = \frac{8}{5}u_{n-1} - u_{n-2} \qquad \left(u_1 = \frac{4}{5} \qquad u_2 = \frac{7}{25}\right)$$

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$$u_n = \frac{1}{2} \left(\frac{4}{5} - \frac{3}{5} \mathbf{i} \right)^n + \frac{1}{2} \left(\frac{4}{5} + \frac{3}{5} \mathbf{i} \right)^n$$

Goal: verify properties of $\langle u_n \rangle_{n\in\mathbb{N}}$ algorithmically

$\mathbf{q} \cdot M^n \qquad n \in \mathbb{N}$

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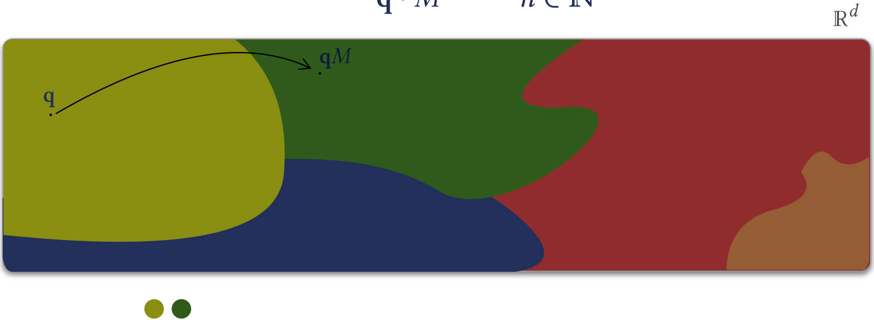

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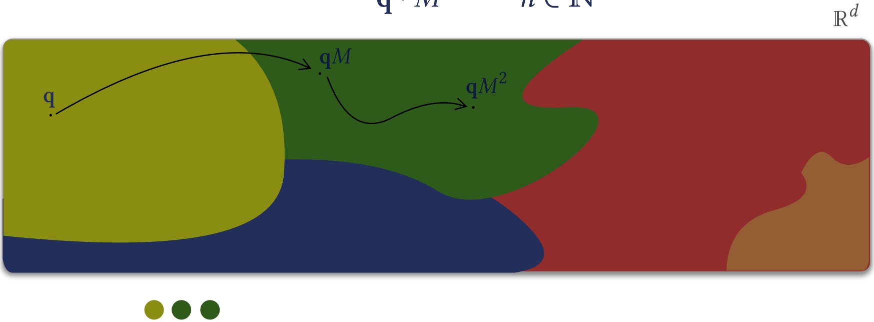

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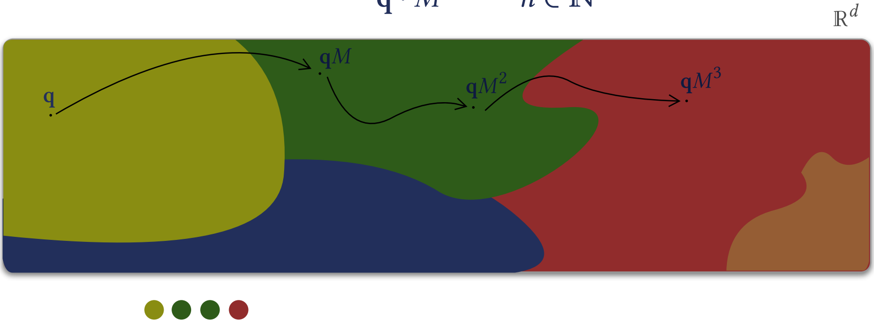

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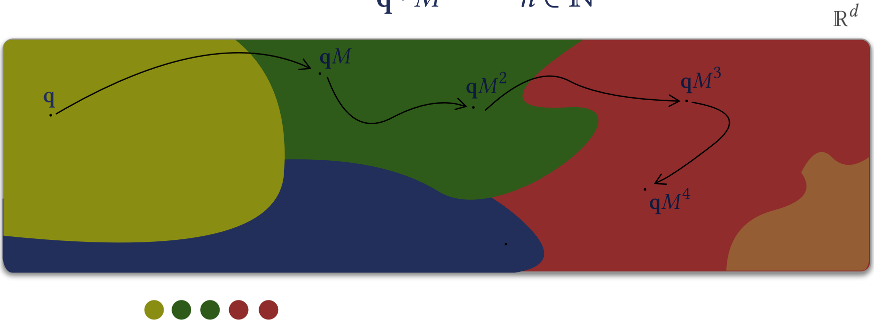

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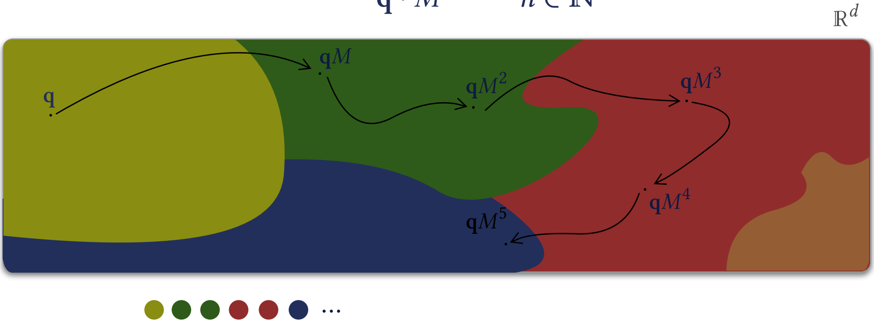

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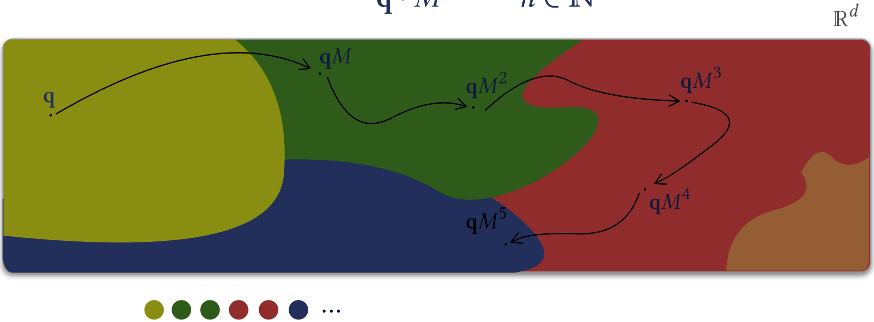

Ideally we would like a procedure for:

INPUT: partition, $\mathbf{q}$, M, $\mathcal{A}$ (Büchi automaton)

OUTPUT: Is ● ● ● ● ● ● … accepted by $\mathcal{A}$?

Theorem

There is a procedure for

diagonalisable   prefix-independent

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$\langle u_n \rangle_{n\in\mathbb{N}}$	2	−1	0	8	−12	0	24	−34	0	⋯
ζ	±	±	0	±	±	0	±	±	0	⋯

What can we say about the word ζ?

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Theorem (Skolem-Mahler-Lech)

ζ is ultimately-periodic i.e:

$$\zeta = w_1 w_2^{\omega}.$$

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THEOREM (SKOLEM-MAHLER-LECH)

ζ is ultimately-periodic i.e:

$$\zeta = w_1 w_2^{\omega}.$$

- w_2 can be computed (Berstel and Mignotte 1976),
- computing w_1 has been open for a while,
- asymptotic behavior is simpler

$\langle u_n \rangle_{n\in\mathbb{N}}$	2	−1	0	8	−12	0	24	−34	0	⋯
σ	+	−	0	+	−	0	+	−	0	⋯

$\langle u_n \rangle_{n\in\mathbb{N}}$	2	-1	0	8	-12	0	24	-34	0	$\cdots$
σ	$+$	$-$	0	$+$	$-$	0	$+$	$-$	0	$\cdots$

Is σ ultimately-periodic as well?

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Is σ ultimately-periodic as well?

counter-example

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{pmatrix}^n \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \cos(n \overbrace{\theta}^{\arg(4/5-3/5\mathbf{i})})$$

Lemma

The sign description σ of diagonalisable sequences is almost-periodic.

$$abbbaababababababbabbbababbabababbbabababbabbbababababbb\cdots$$

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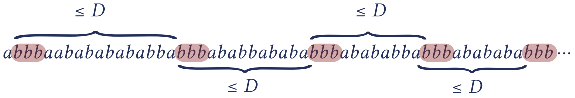
Almost-periodic words



Definition

An infinite word $\alpha \in \Sigma^\omega$ is *almost-periodic* if for every word $w \in \Sigma^*$, there exists $p \in \mathbb{N}$ such that either:

- w does not occur in α after position p, or
- w occurs in every factor of α of length p.

Synonyms: uniformly recurrent sequence, minimal sequence.
Examples: ultimately-periodic words, Sturmian words, Thue-Morse
Non-example: $aba^2ba^3ba^4b\cdots$.

Theorem (Semenov 1984)

MSO *theory of* $(\mathbb{N}, >, P)$ *is decidable when* P *is effectively almost-periodic.*

What is so desirable about almost-periodic words?

Theorem (Semenov 1984)

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There is a procedure for:

INPUT: $\mathcal{A}$ (Büchi automaton)

OUTPUT: Is  accepted by $\mathcal{A}$?

How do we prove the main lemma?

Lemma

The sign description σ of diagonalisable sequences is almost-periodic.

$\langle u_n \rangle_{n \in \mathbb{N}}$	-2	4	-8	-16	-32	12	-20	$\cdots$
σ	$-$	$+$	$-$	$-$	$-$	$+$	$-$	$\cdots$

$\leq D$ $\quad$ $\leq D$

1. Reduce to a much simpler linear recurrence sequence
2. Use a compactness argument to derive the bound

Lemma

The sign description σ of diagonalisable sequences is almost-periodic.

First

$$\overline{\mathbb{Q}} \quad \overline{\mathbb{Q}}$$

$$u_n = c_1\Lambda_1^n + \cdots + c_d\Lambda_d^n \qquad |\Lambda_1| \geq |\Lambda_2| \geq \cdots \geq |\Lambda_d|$$

$$c_1\Lambda_1^n + \cdots + c_r\Lambda_r^n + c_{r+1}\Lambda_{r+1}^n + \cdots + c_d\Lambda_d^n \qquad |\Lambda_1| = \cdots = |\Lambda_r| > |\Lambda_{r+1}|$$

$D(n)$ $\qquad$ $R(n)$

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Proposition.

$$\exists n_0 \ \forall n > n_0 \quad |D(n)| > |R(n)|.$$

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$$u_n = c_1\Lambda_1^n + \cdots + c_d\Lambda_d^n \qquad c_1, c_d \in \overline{\mathbb{Q}}$$

$$|\Lambda_1| \geq |\Lambda_2| \geq \cdots \geq |\Lambda_d|$$

$$\underbrace{c_1\Lambda_1^n + \cdots + c_r\Lambda_r^n}_{D(n)} + \underbrace{c_{r+1}\Lambda_{r+1}^n + \cdots + c_d\Lambda_d^n}_{R(n)}$$

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$$\cos(n\theta) + 2^{-n}$$

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Proof. Using bounds on sums of S-units. p-adic subspace theorem.

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Not effective! reason for "prefix-independent" $\cos(n\theta) + 2^{-n}$

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We have reduced to analysing the sign of

$$v_n = c_1\lambda_1^n + \cdots + c_r\lambda_r^n, \qquad (|\lambda_1| = \cdots = |\lambda_r| = 1)$$

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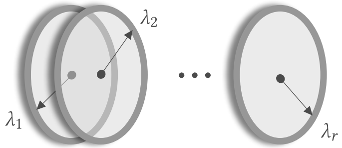

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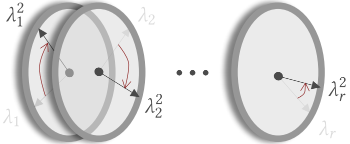

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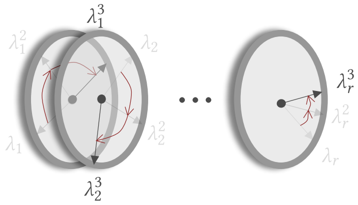

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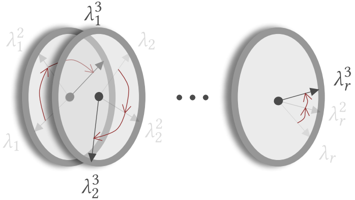

There is a subset $\mathbb{T}_\lambda \subseteq \mathbb{T}^r$ that is:

- compact,
- semialgebraic,
- can effectively be constructed, and
- $\left\{ (\lambda_1^n, \ldots, \lambda_r^n) \ : \ n \in \mathbb{N} \right\}$ is dense in $\mathbb{T}_\lambda$ - small over-approximation

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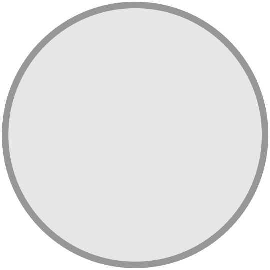
$$f \;:\; \mathbb{T}_\lambda \to \mathbb{R} \qquad (x_1, \ldots, x_r) \mapsto (c_1x_1, \ldots, c_rx_r)$$

$\mathbb{T}_\lambda$



Almost-periodic σ, proof

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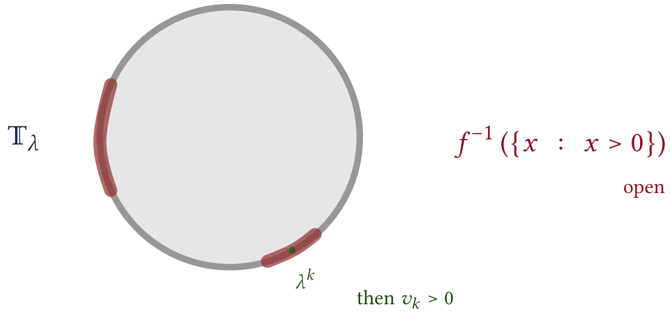

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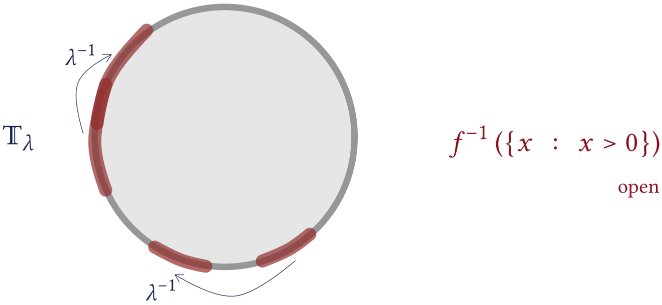

in the next time-step

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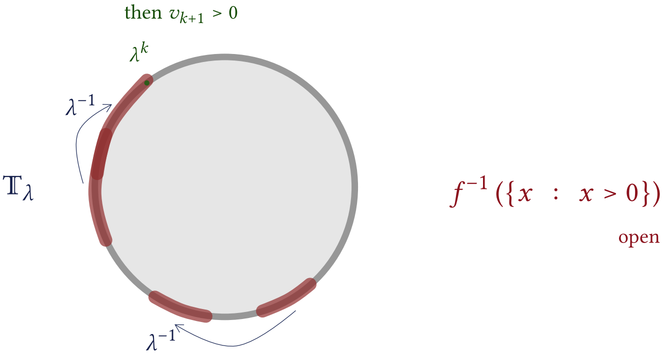

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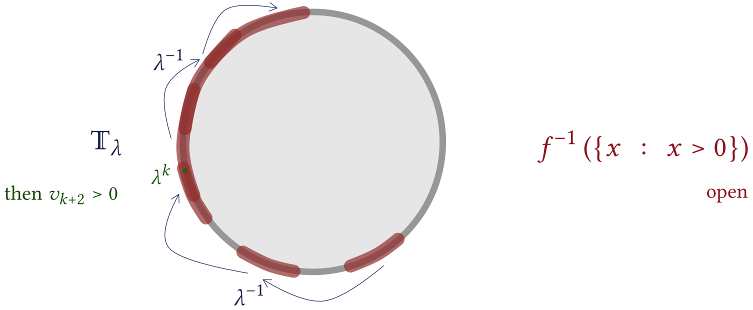

in two time-steps

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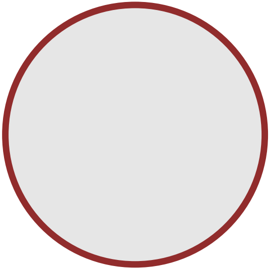
$\mathbb{T}_\lambda$



$f^{-1}(\{x \;:\; x > 0\})$

open

density $\implies$ $\left\{\lambda^{-n}P \;:\; n \in \mathbb{N}\right\}$ is an open cover of $\mathbb{T}_\lambda$

compactness $\implies$ admits a finite sub-cover

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- Words with 0 need more care (we have to apply Skolem's theorem)

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- Words with 0 need more care (we have to apply Skolem's theorem)

We want to decide: given $\mathcal{A}$, does $\mathcal{A}$ accept σ.

For this we need to be able to compute some things about σ.

Furthermore, (since T_λ is semialgebraic and c_i, λ_i are algebraic) by manipulating $\mathrm{FO}(\mathbb{R}, >, +, \cdot, 0, 1)$ formulas we can:

- decide whether a word w occurs infinitely often in σ,
- if it does, compute a bound on the distance between consecutive occurrences.

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- if it does, compute a bound on the distance between consecutive occurrences.

Define a subroutine:

$$\mathrm{inter}_\sigma \;:\; \Sigma^* \to \{\mathbf{no}\} \cup \mathcal{P}(\Sigma^*)$$

$$w \mapsto \begin{cases} \mathbf{no} & \text{if } w \text{ does not occur i.o. in } \sigma \\ \{w_1, \dots, w_k\} & \text{otherwise} \end{cases}$$

$$\sigma = v\ w\ w_3\ w\ w_1\ w\ w_{k-1}\ \cdots$$

Theorem

The following problem is decidable:

INPUT: $\mathcal{A}$ *(deterministic and prefix-independent Müller automaton),* inter_σ

OUTPUT: *Does* $\mathcal{A}$ *accept* σ*?*

```
w:=a
```
(for some $a \in \Sigma$ for which $\text{inter}_\sigma(a) \neq$ **no**)

```
while(true):
```

$\{w_1, \dots, w_k\} := \text{inter}_\sigma(w)$

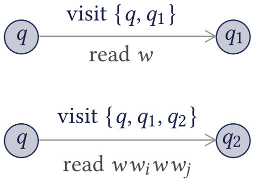
`if` $\exists q \in Q$ and w_i, w_j such that we see *more* states from q with $w w_i w w_j$ than we do from q with w

`then`

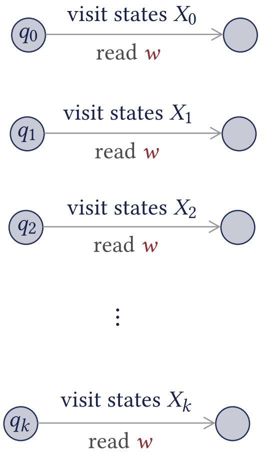
$w := w w_i w w_j$

`else`

`break`

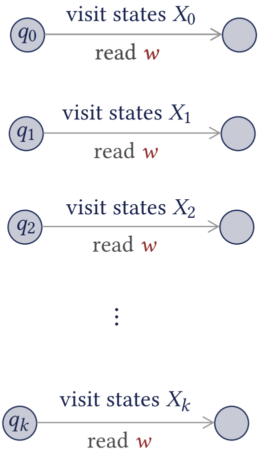


$\{q, q_1, q_2\} \supset \{q, q_1\}$



Choose $q := q_i$ such that $X := X_i$ has minimal cardinality among $X_0, X_1, \dots, X_k$

return **yes** if and only if X is a final set of states



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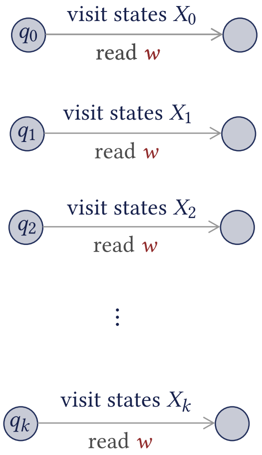
Correctness.

$$\sigma = p\ w\ w_2\ w\ w_4\ w\ w_1\ w\ \cdots$$



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return **yes** if and only if X is a final set of states

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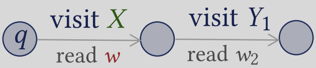

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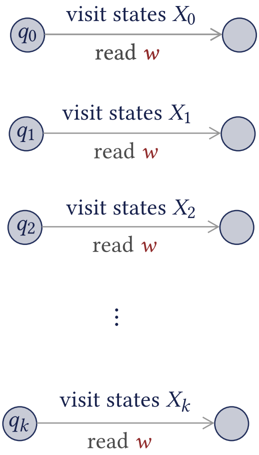
(cannot see more states)



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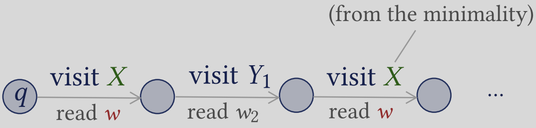

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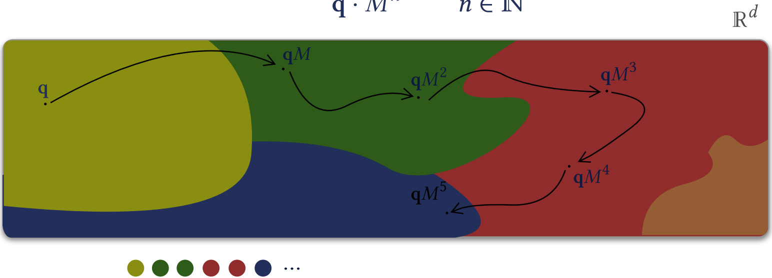
Some suffix of σ is accepted by $\mathcal{A}$ started in state q

Some suffix of σ is accepted by $\mathcal{A}$ started in state q

implies

σ is accepted by $\mathcal{A}$ (because $\mathcal{A}$ is prefix-independent)

$\mathbf{q} \cdot M^n \qquad n \in \mathbb{N}$



THEOREM

There is a procedure for diagonalisable prefix-independent

INPUT: semialgebraic partition, q, M, $\mathcal{A}$ (Büchi automaton)

OUTPUT: Is ● ● ● ● ● ● … accepted by $\mathcal{A}$?

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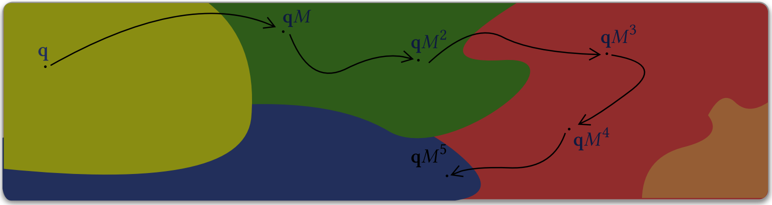

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LRS are closed under component-wise product and addititon

(Presented 1 dimensional case only)

Context

- For general LRS, complications are witnessed by the fact that σ for a non-diagonalisable LRS is not necessarily almost-periodic

  (a decision procedure for "+ i.o" can be used to compute Lagrange constants)

- This positive result was known for the formula "+ i.o"

  (Ouaknine, Worrell 2014)

- Since a Markov chain is a LRS, it has some implications for logics presented on: (Agrawal, Akshay, Genest, Thiagarajan 2015), (Beauquier, Rabinovich, Slissenko, 2006)

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```
while(true):
   x  =  3x+2y;
   y  =  -15z;
   z  =  2x;
   if(15x^2-2x>3y)  a++;
   if(-2z+12y<0)    b++;
```

Is $a_n = O(b_n)$?

Thank you