The Density of Positive Entries of a Linear Recurrence

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I. The Problem

II. The Theorems

III. The Example, or First Observation

IV. The Proof

V. The Open Problem

1. The Problem

$x \leftarrow 0$; $y \leftarrow 6$; $z \leftarrow 4$

```
while true do
    x ← 4x + 3y
    y ← 4y − 3x
    z ← 5z
    if y + z > 0 then
      Region A
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  else
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- arbitrary number of variables ranging over integers
- linear updates
- polynomial inequality

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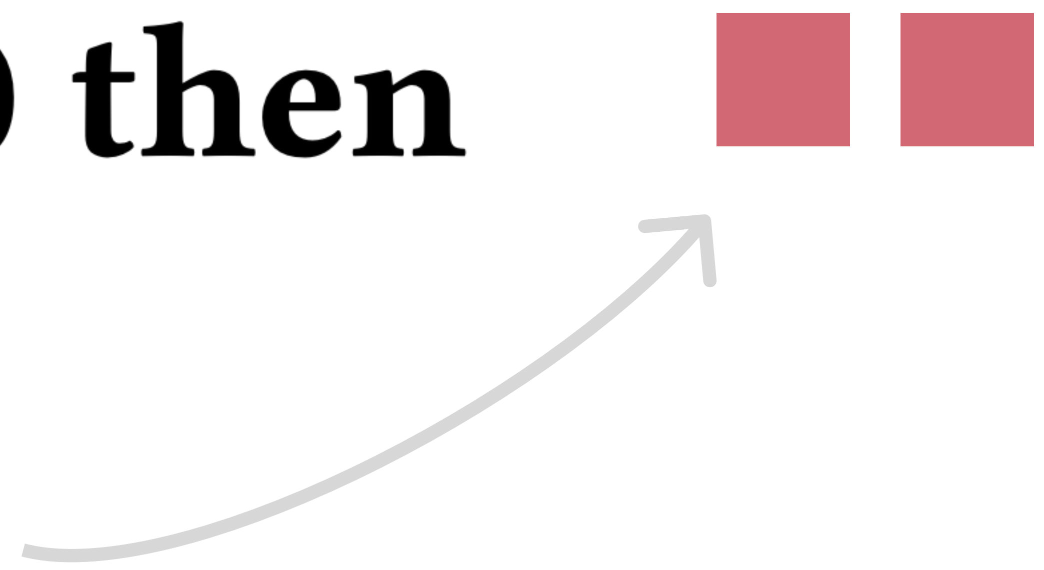

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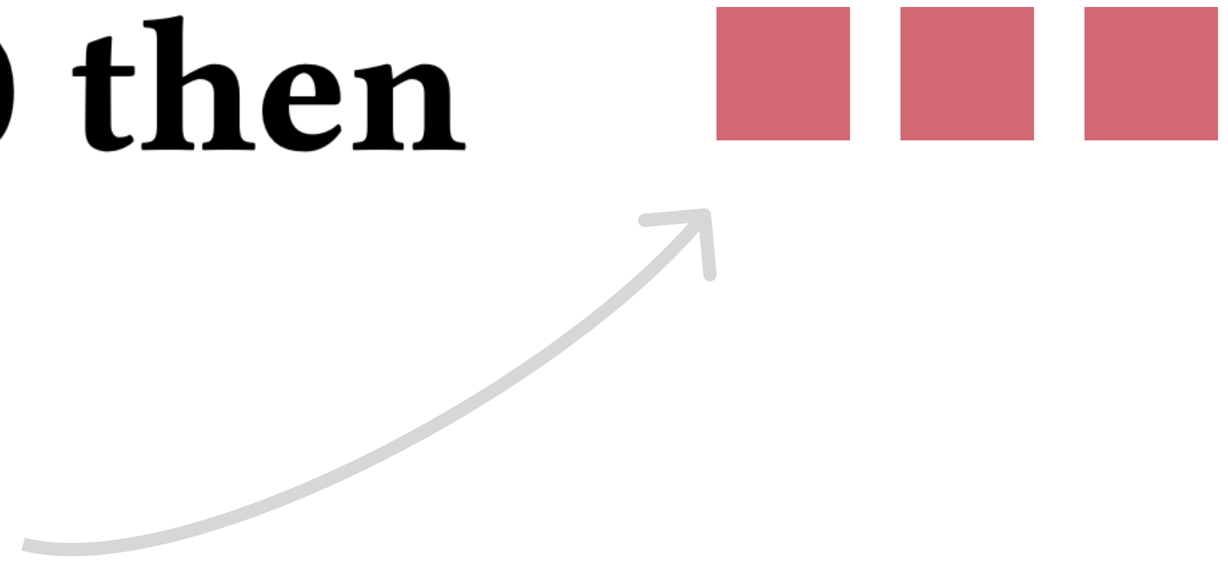

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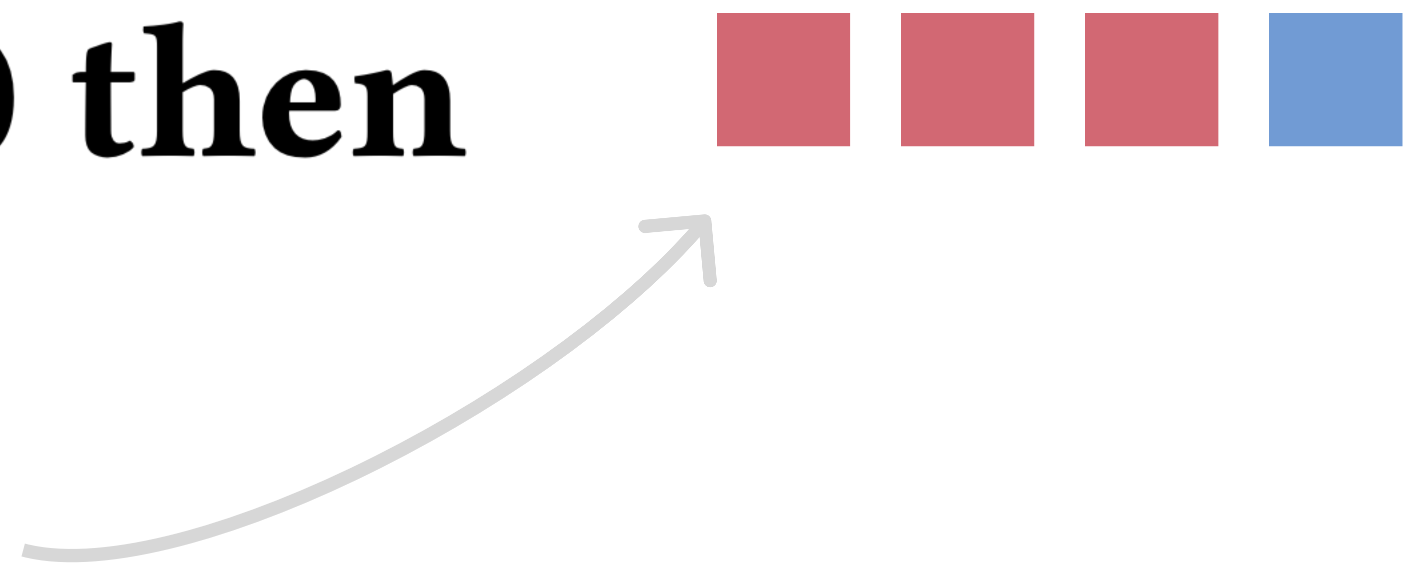

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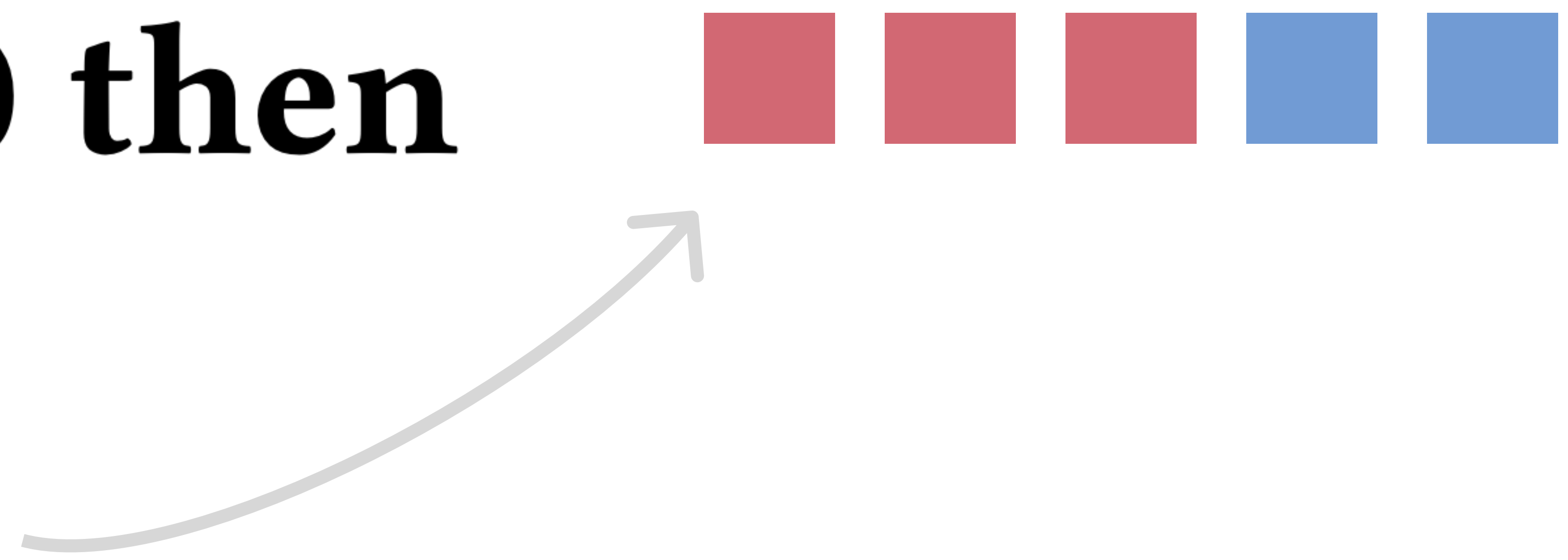

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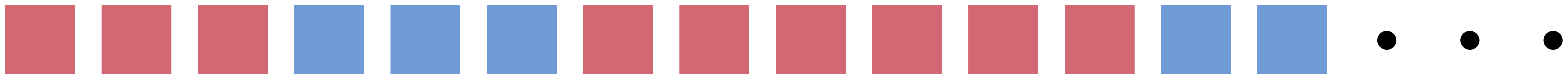

Decision questions:

1. Is Region A reached?
   (Is there at least one ■ ?)

2. Is Region A reached infinitely often?
   (Are there infinitely many ■ ?)

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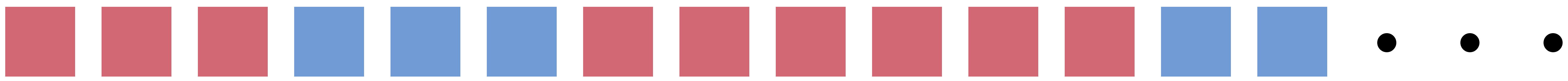

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In this paper:

3. How much more frequent are ■ compared to ■?

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Set of ■

1. Is it empty?
2. Is it infinite?
3. How big is it inside $\mathbb{N}$?

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Density

$S \subset \mathbb{N}$

$$\mathcal{D}_\circ(S) := \liminf_{n\to\infty} \frac{|\{1,2,\ldots,n\} \cap S|}{n}$$

What proportion of first n numbers are in S ($n \to \infty$) ?

Examples:

- $\mathcal{D}_\circ(\text{ Finite Set}) = 0$, $\mathcal{D}_\circ(\text{ co-Finite Set}) = 1$
- $\mathcal{D}_\circ(\text{Arithmetic progression of length } k) = 1/k,$
- $\mathcal{D}_\circ(\text{Geometric progression}) = 0,$
- $\mathcal{D}_\circ(\text{Primes}) = 0,$ (Prime Number Theorem).

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What proportion of first n numbers are in S ($n \to \infty$) ?

$$\mathcal{D}_\circ(X) = 0 \Rightarrow X \text{ is sparse}$$
$$\mathcal{D}_\circ(X) = 1 \Rightarrow X \text{ is very dense}$$

if $\mathcal{D}_\circ(X) = \mathcal{D}^\circ(X)$, denote by $\mathcal{D}(X)$

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$$\frac{\text{Number of } \blacksquare \text{ in first } n \text{ entries}}{n}, \quad n \in \mathbb{N}$$

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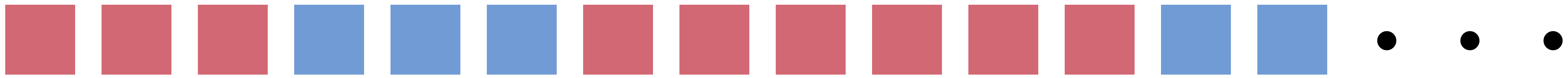

$$\lim_{n\to\infty} \frac{\text{Number of } \blacksquare \text{ in first } n \text{ entries}}{n}$$

Exists due to:

Jason P Bell and Stefan Gerhold. On the positivity set of a linear recurrence sequence. *Israel Journal of Mathematics*, 157(1):333–345, 2007.

Denote it by $\mathcal{D}$.

Main character of the story

II. The Theorems

Theorems

Theorem 1. "$\mathcal{D} = 0?$" is decidable.

(so is "$\mathcal{D} = 1?$" by symmetry)

Theorem 1a. For diagonalisable update matrices:

$$\mathcal{D} = 0 \Leftrightarrow \text{finitely many} \ \blacksquare$$

Theorem 2. $\mathcal{D}$ can be computed to arbitrary precision.

Theorem 3. "$\mathcal{D} \in \mathbb{Q}?$" is decidable,
when there are at most three dominant eigenvalues.

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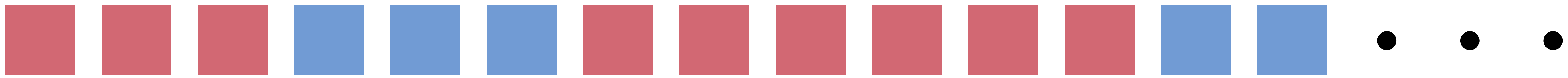

To Summarise:

- We can decide if Region A is entered not too rarely
- We can say a lot about the asymptotic frequency of entering Region A/Region B

III. The Example, or First Observation

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How frequently is Region A entered?

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How frequently is Region A entered?

$$0.732279\cdots = \frac{\cos^{-1}(-2/3)}{\pi}$$

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```

$$\underbrace{\begin{pmatrix} 0 & 6 & 4 \end{pmatrix} \cdot \begin{pmatrix} 4 & -3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}^{n} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$$

y+z in the nth loop iteration

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$$5^n \begin{pmatrix} 4/5 & -3/5 & 0 \\ 3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}^n$$

2D rotation

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2D rotation

$$r(\cos\theta, \sin\theta) \cdot \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = r(\cos\theta\cos\varphi - \sin\theta\sin\varphi, \cos\theta\sin\varphi + \sin\theta\cos\varphi)$$

$$= r(\cos(\theta+\varphi), \sin(\theta+\varphi))$$

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$$5^n \begin{pmatrix}4/5 & -3/5 & 0\\ 3/5 & 4/5 & 0\\ 0 & 0 & 1\end{pmatrix}^n = 5^n \begin{pmatrix}\cos n\varphi & -\sin n\varphi & 0\\ \sin n\varphi & \cos n\varphi & 0\\ 0 & 0 & 1\end{pmatrix}$$

Rotation in the first two coordinates by $\varphi = \cos^{-1} 4/5$

$x \leftarrow 0;\ y \leftarrow 6;\ z \leftarrow 4$
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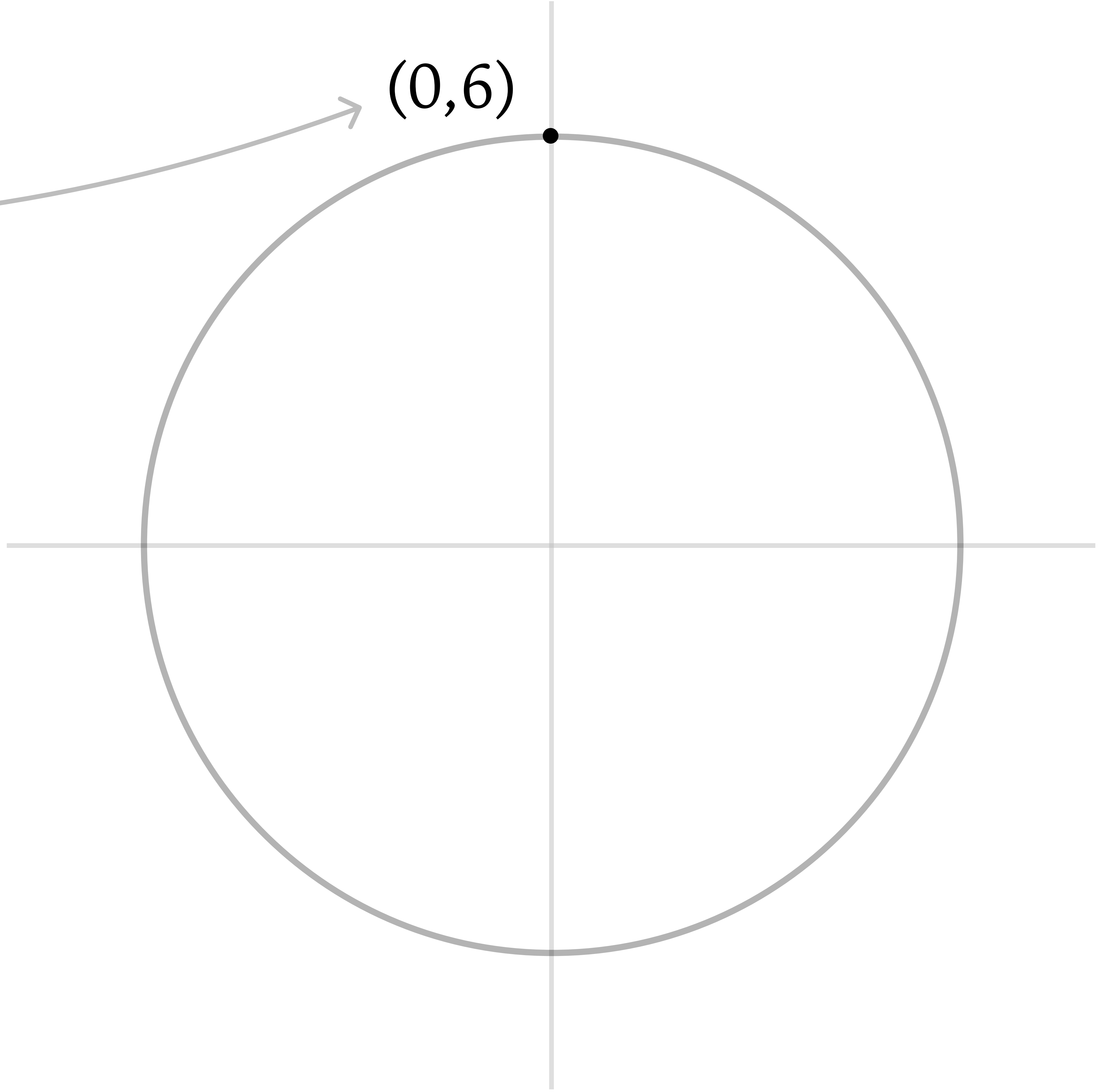

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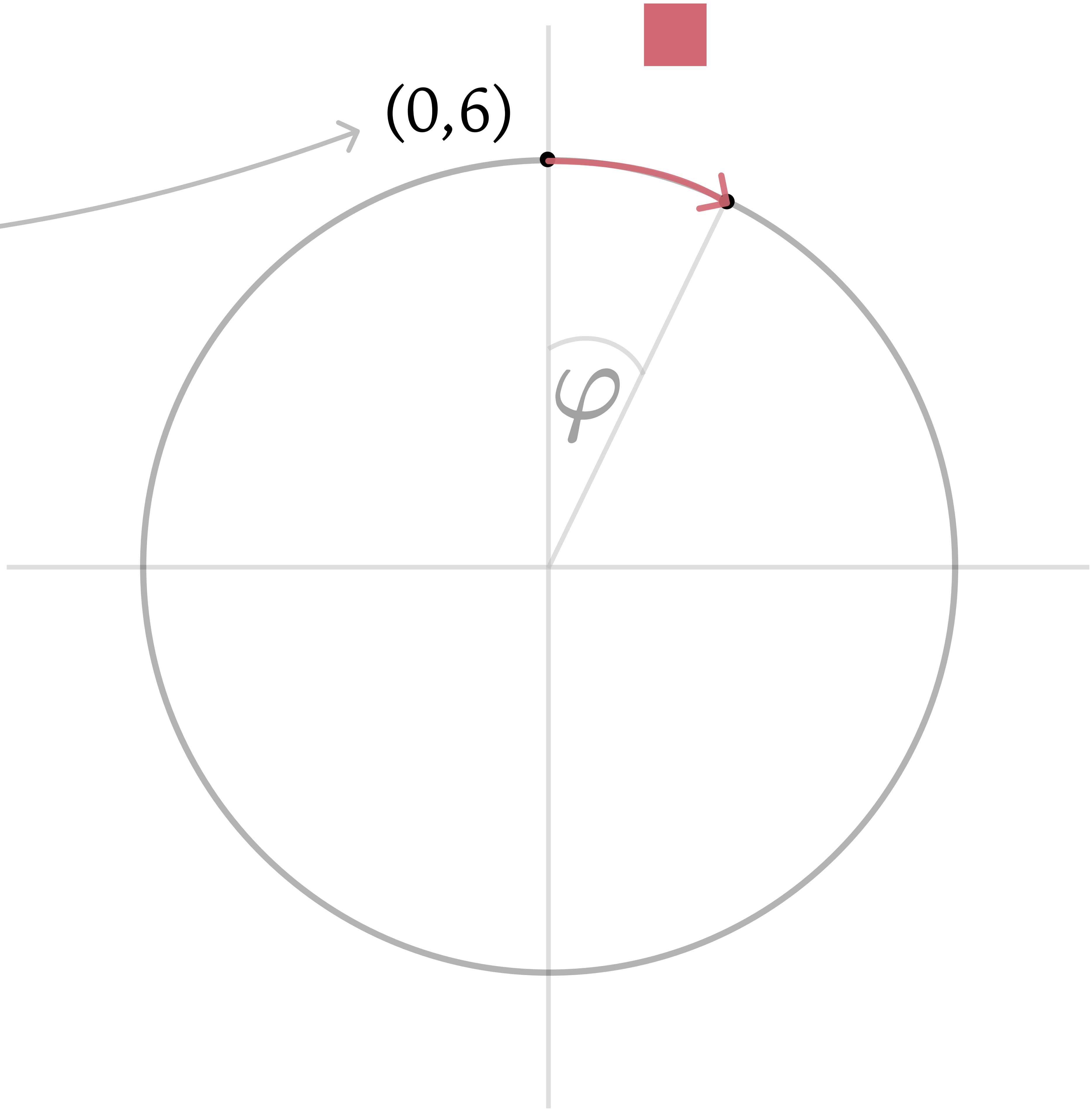

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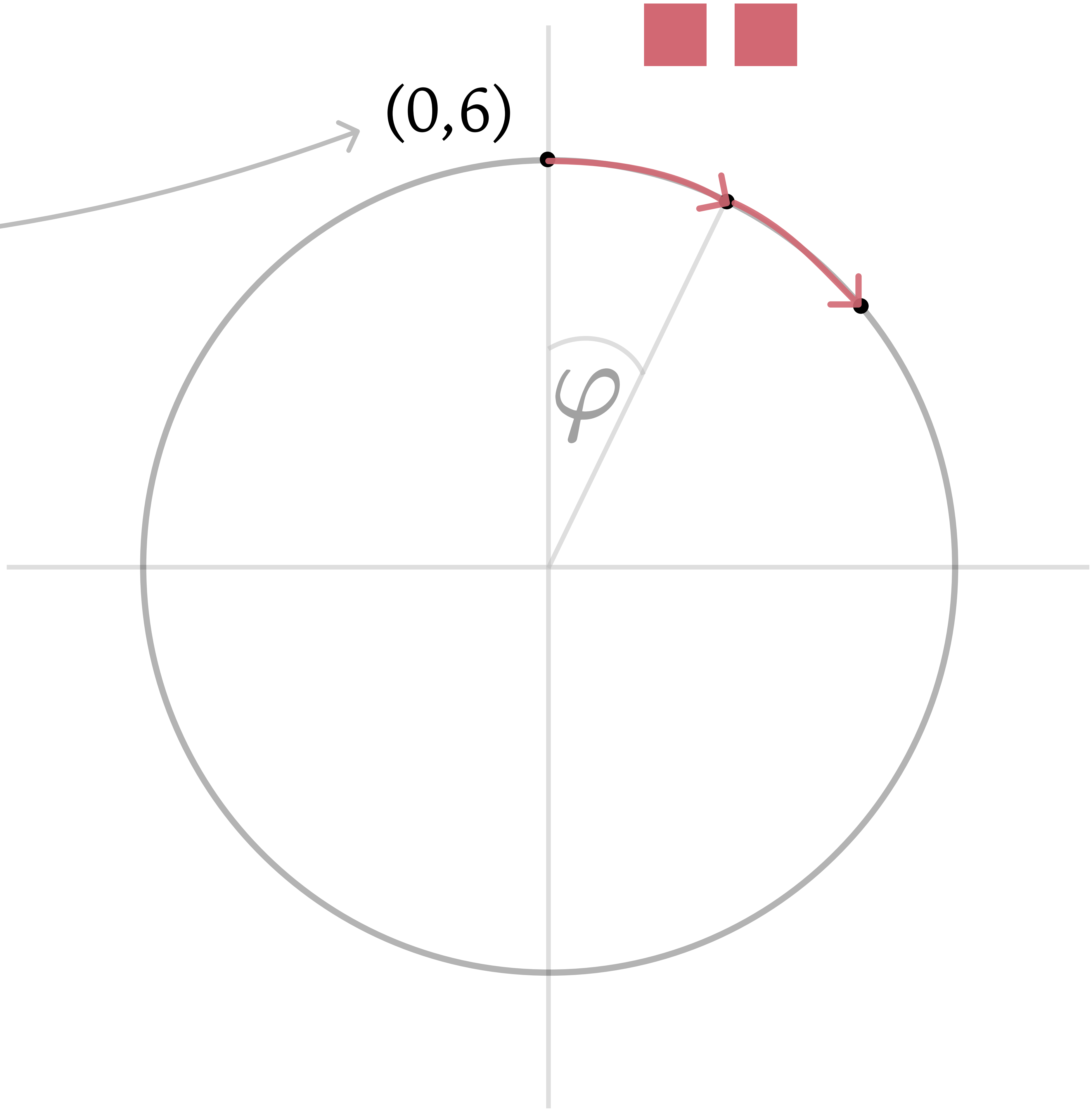

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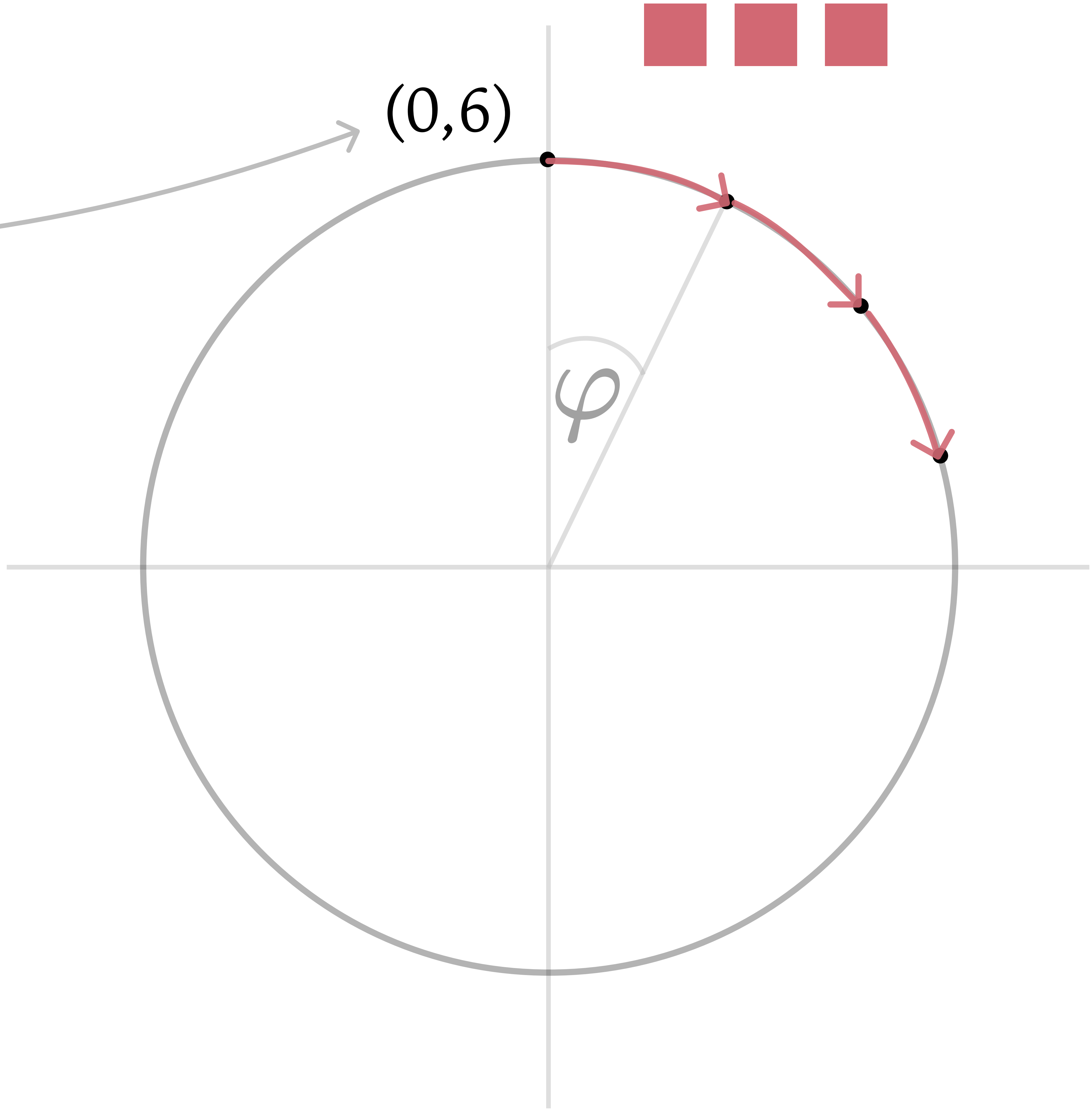

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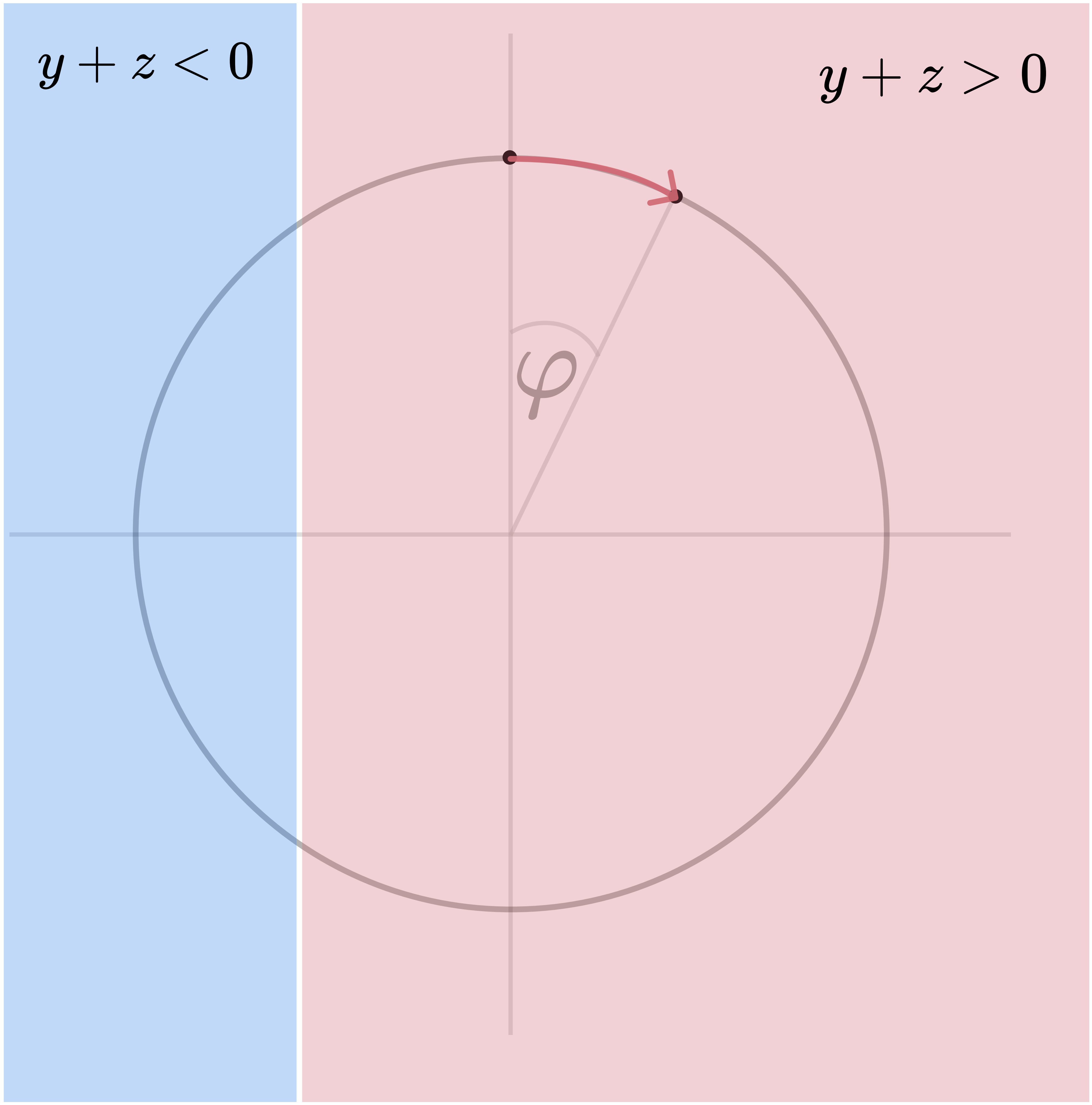

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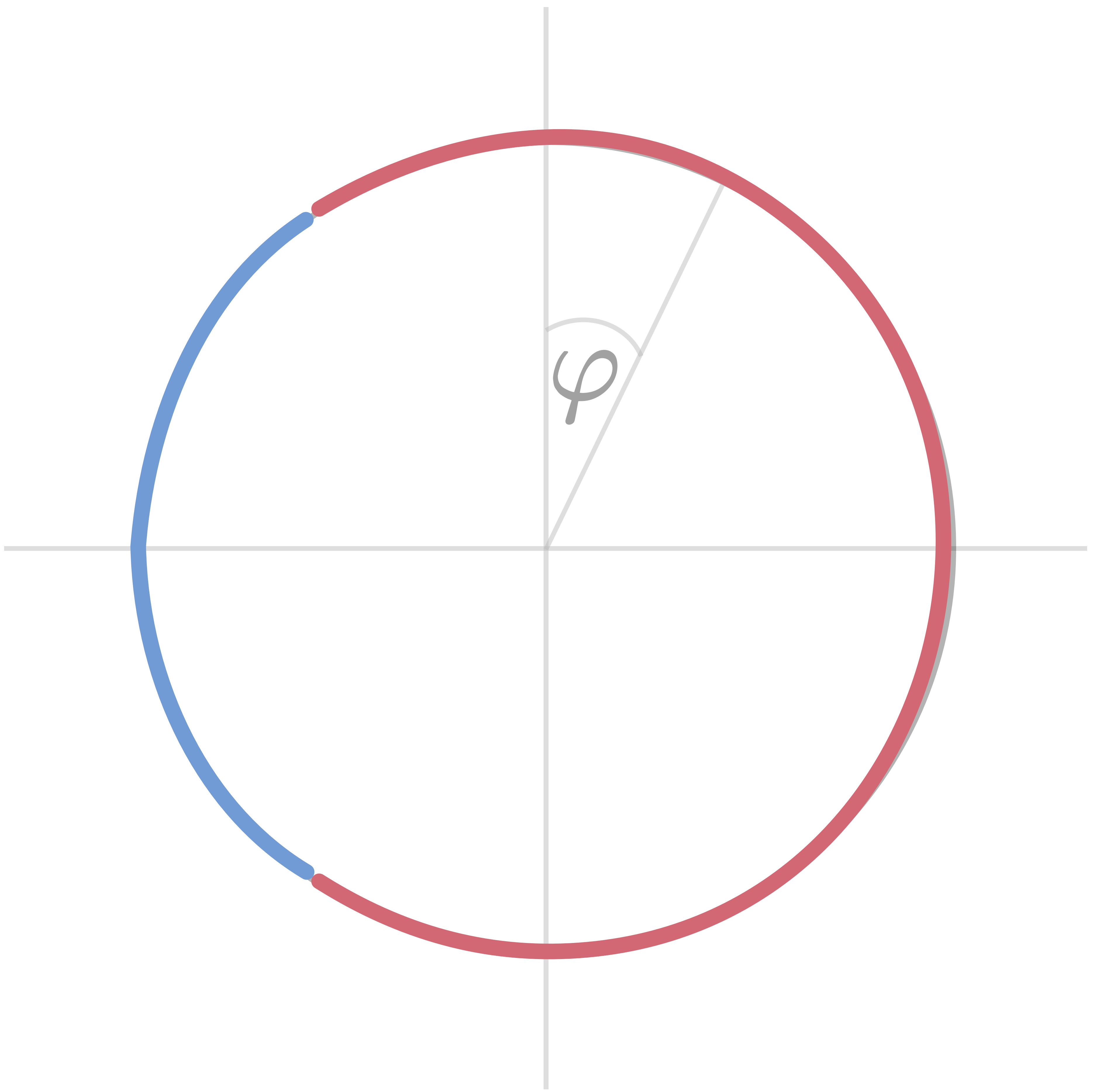
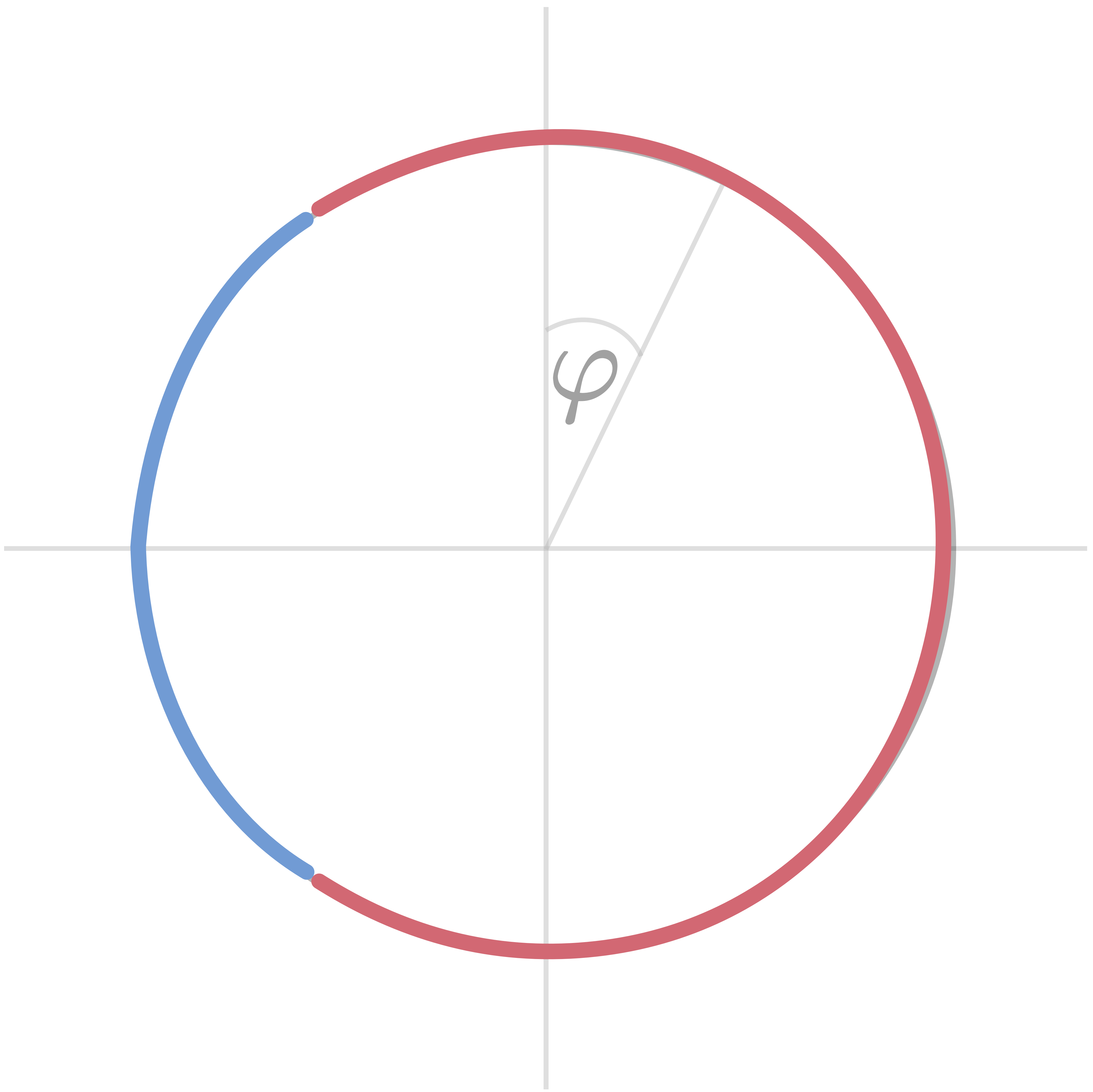


How frequently are we on the red arc?

$\varphi = \cos^{-1} 4/5$



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Theorem (Weyl, 1910). Let ρ be an irrational real number. Then the sequence:

$$\rho,\ 2\rho,\ 3\rho,\ \ldots$$

is uniformly distributed mod 1.

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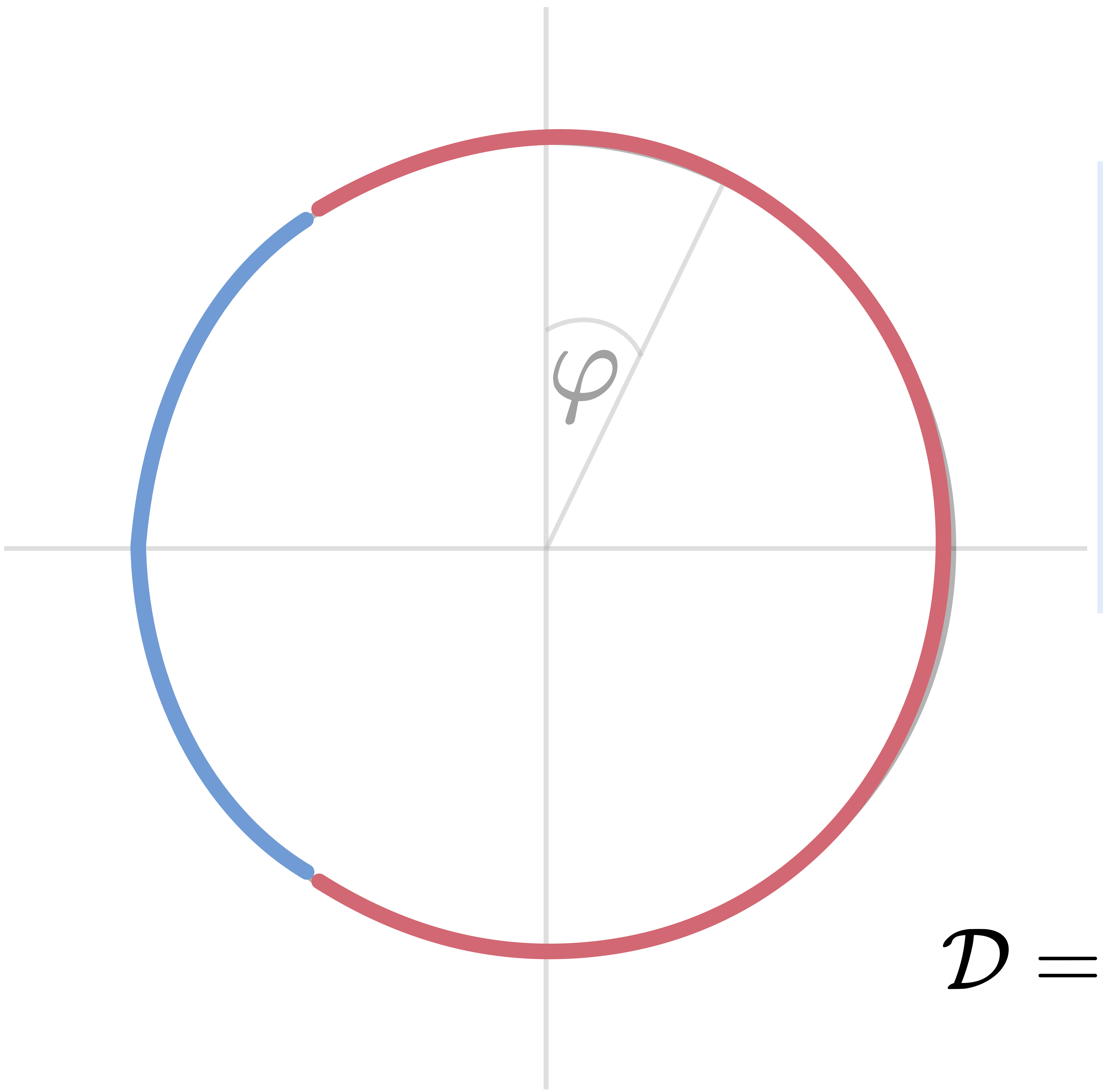

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is uniformly distributed mod 1.

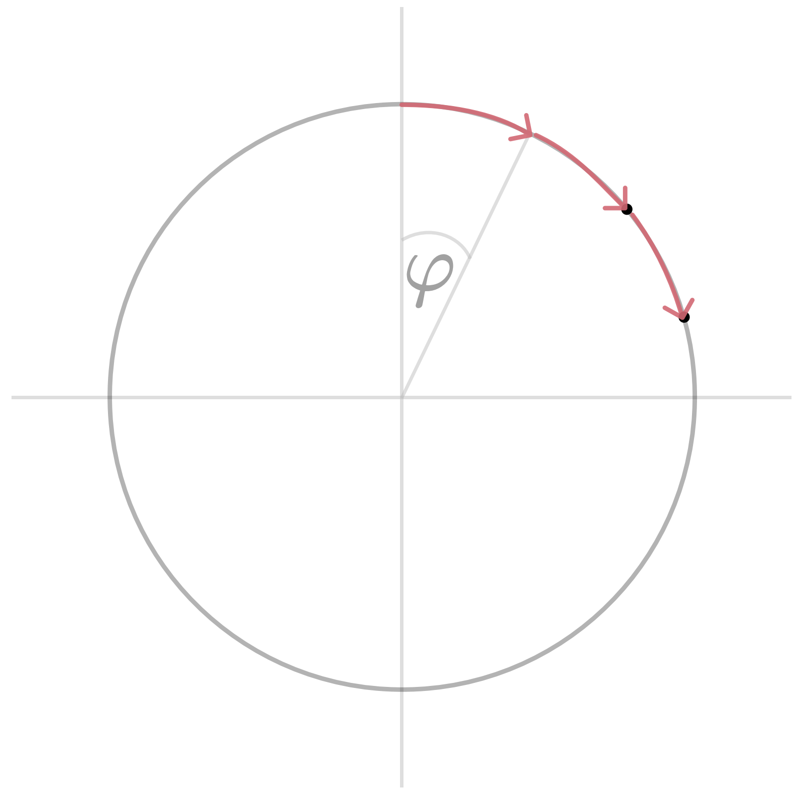
$$\mathcal{D} = \frac{\text{length of —}}{2\pi} = \frac{\cos^{-1}(-2/3)}{\pi} = 0.732278\ldots$$

Scheme

Conserve sign information

$$\underbrace{\begin{pmatrix}4 & -3 & 0\\ 3 & 4 & 0\\ 0 & 0 & 1\end{pmatrix}^{n}}_{\text{Update Matrix}}$$

Rotations on the circle



Weyl's equidistribution

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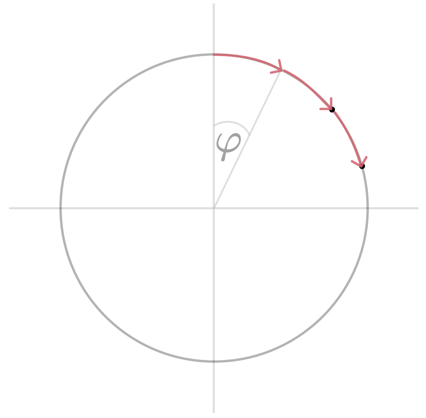

Weyl's equidistribution

$$\underbrace{\mathbf{M}}_{\text{General Update Matrix}}$$

?

?

?

Scheme

$$\underbrace{\begin{pmatrix} 4 & -3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{Update Matrix}}{}^{n}$$

Conserve sign information

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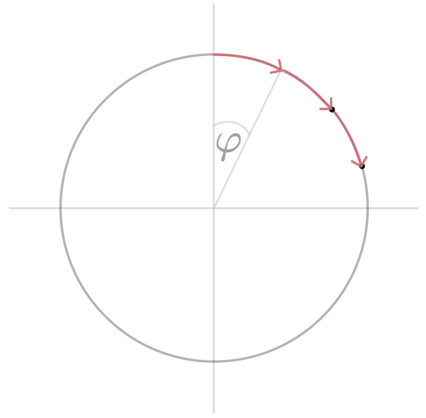


Weyl's equidistribution

$$\underbrace{\mathbf{M}}_{\text{General Update Matrix}}$$

Conserve enough information for density

Rotations on a subgroup of $\mathbb{T}^n$

A stronger version of Weyl's theorem

$+$

Koiran's approximation of volumes of definable sets

IV. The Proof

Linear Recurrence Sequences

characteristic polynomial of M

Theorem (Cayley-Hamilton). $f(\underbrace{M}) = \mathbf{0}$

matrix

Linear Recurrence Sequences

characteristic polynomial of M

Theorem (Cayley-Hamilton). f(M) = **0**

matrix

$$2M^3 - 4M^2 + M + 5I = \mathbf{0}$$

Linear Recurrence Sequences

characteristic polynomial of M

Theorem (Cayley-Hamilton). f(M) = **0**

matrix

$$2M^{n+3} - 4M^{n+2} + M^{n+1} + 5M^{n} = \mathbf{0}$$

Linear Recurrence Sequences

characteristic polynomial of M

Theorem (Cayley-Hamilton). f(M) = **0**

matrix

$$2M^{n+3} - 4M^{n+2} + M^{n+1} + 5M^n = \mathbf{0}$$

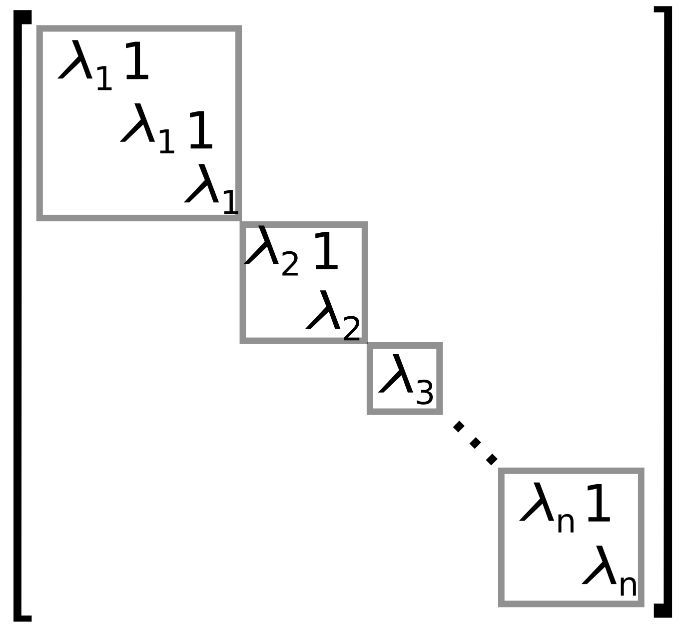
$$u_n := (M^n)_{i,j}$$

$$2u_{n+3} - 4u_{n+2} + u_{n+1} + 5u_n = 0$$

Entries of M^n are LRS.

We are interested in the density of $\{n : u_n > 0\}$

Linear Recurrence Sequences



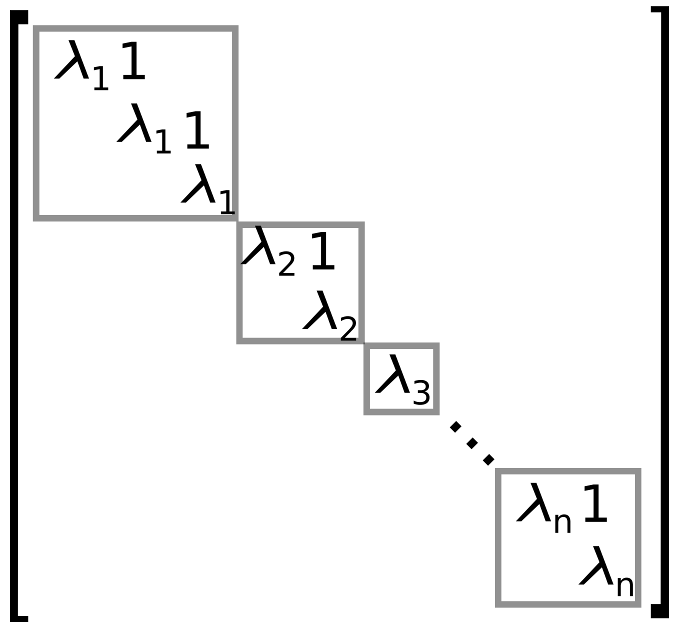
By Jordan decomposition

$$u_n = \sum_{i=1}^{d} P_i(n) \; \lambda_i^n$$

$\lambda_i \in \bar{\mathbb{Q}}$

$P_i \in \bar{\mathbb{Q}}[x]$

Linear Recurrence Sequences



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$\lambda_i \in \overline{\mathbb{Q}}$

$P_i \in \overline{\mathbb{Q}}[x]$

$$\mathcal{D} := \text{density of } \{n\ :\ u_n > 0\}$$

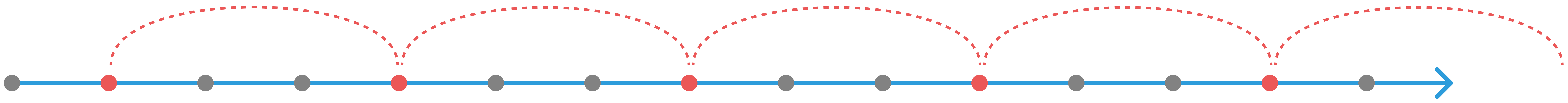
positivity set of $(u_n)_{n\in\mathbb{N}}$

Preprocessing

Split the problem: $(u_{nT+l})_{n\in\mathbb{N}}, \qquad l \in \{0, 1, \ldots, T-1\}$

compute some large period T

where the smaller problems (i.e. subsequences) have some good properties



Preprocessing

To understand the *good property* of the subsequences, note:

$$u_n = \sum_{i=1}^{d} P_i(n) \ \ \lambda_i^n$$

sometimes there may be multiplicative relations among λ:
integers $z_1, \ldots, z_d$ such that:

$$\lambda_1^{z_1} \lambda_2^{z_2} \cdots \lambda_d^{z_d} = 1$$

so e.g. $\lambda_d^{z_d}$ can be written as a product of integer powers of other roots

In the subsequences all the dependencies are gathered

Preprocessing

Good property: the subsequences have the same sign as

$$\underbrace{\sum_{i\in I} c_i \alpha_i^n}_{\text{independent}} + \underbrace{\sum_{i\in D} c_i \prod_{j\in I} \alpha_j^{q_{i,j}}}_{\text{dependent}} + c + \underbrace{R(n)}_{\text{remainder tends to zero}}, \qquad n \in \mathbb{N}$$

- $\alpha_i \in \bar{\mathbb{Q}}, \quad |\alpha_i| = 1$
- $c_i \in \bar{\mathbb{Q}}$
- $q_{i,j} \in \mathbb{Q}$

Furthermore, the group

$$\{(z_1, \ldots, z_k) \in \mathbb{Z}^k \; : \; \underbrace{\alpha_1^{z_1} \alpha_2^{z_2} \cdots \alpha_k^{z_k}}_{\text{independent}} = 1\} = \{\mathbf{0}\}$$

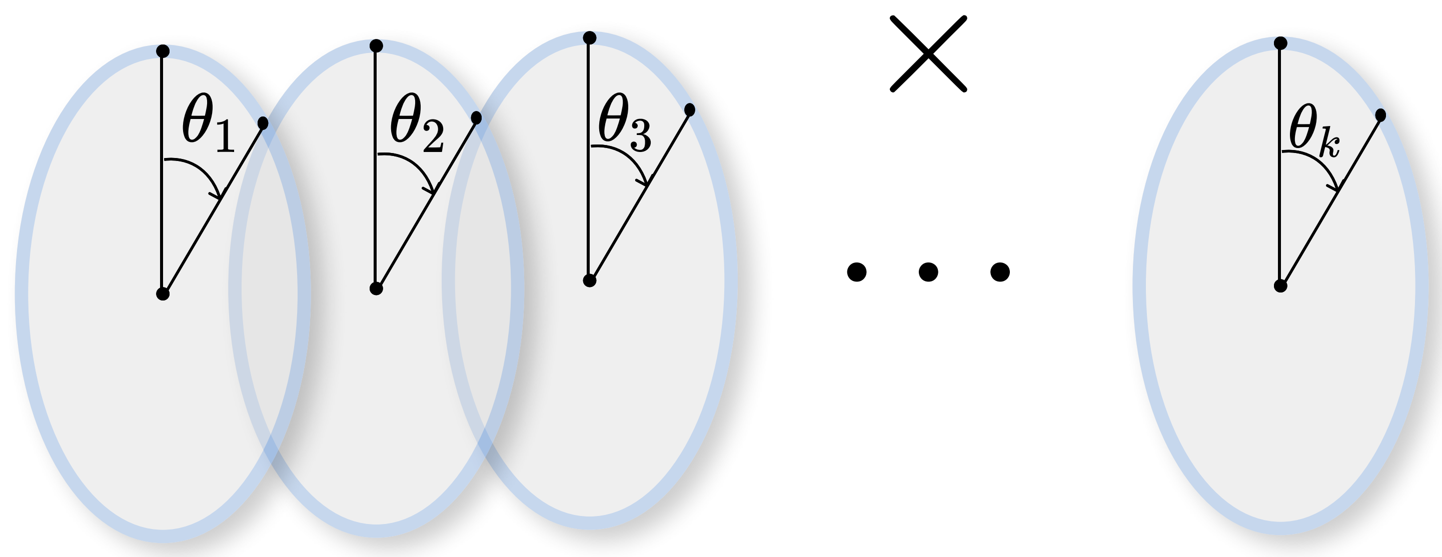
is trivial.

Rotations

$$\alpha_i = e^{2\pi \mathbf{i}\, \theta_i}$$

$$(\alpha_1^n, \alpha_2^n, \ldots, \alpha_k^n), \quad n \in \mathbb{N}$$

$\{\theta_1, \ldots, \theta_k, 1\}$ are linearly independent over $\mathbb{Q}$



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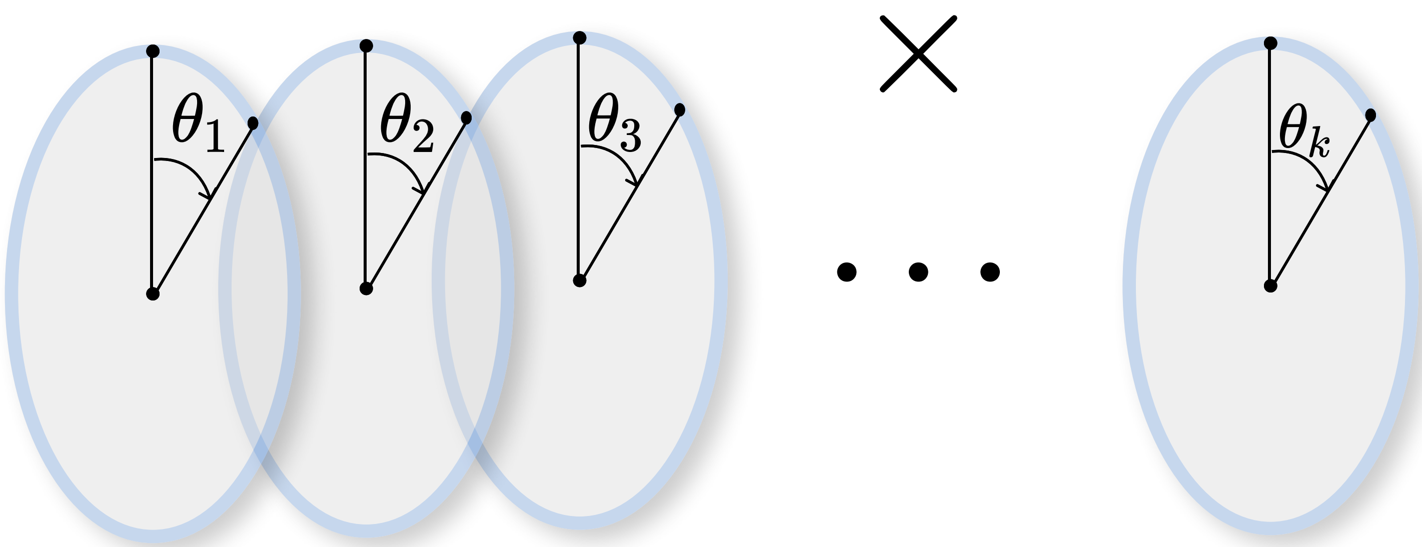
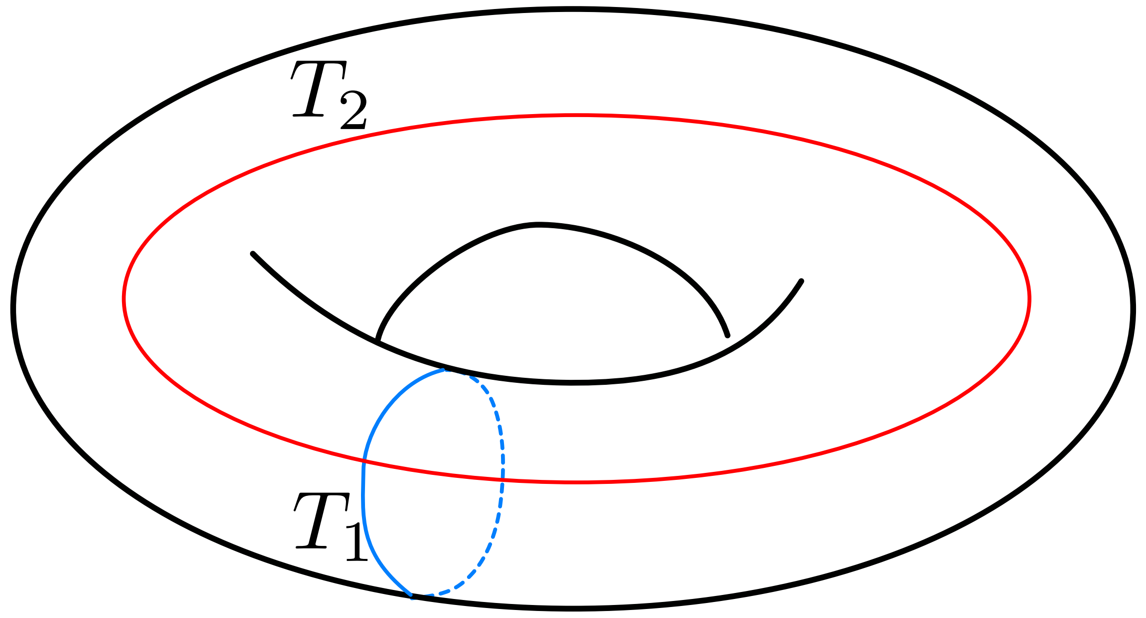
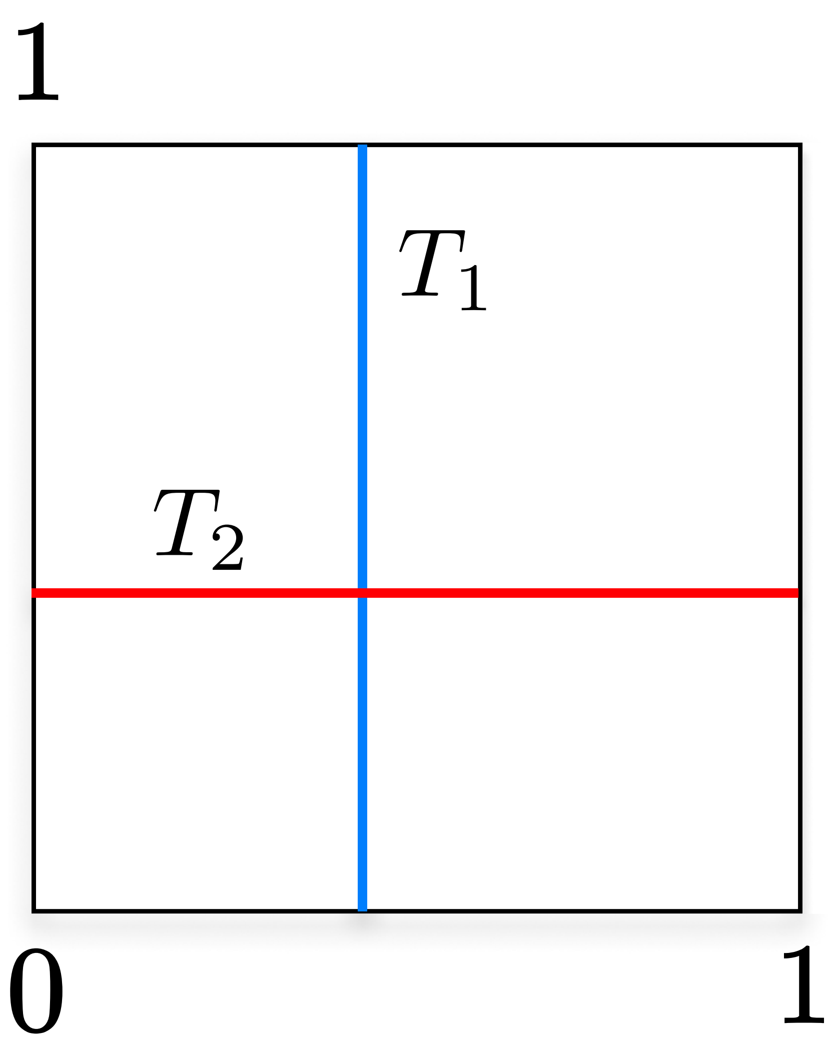


E.g. for k=2





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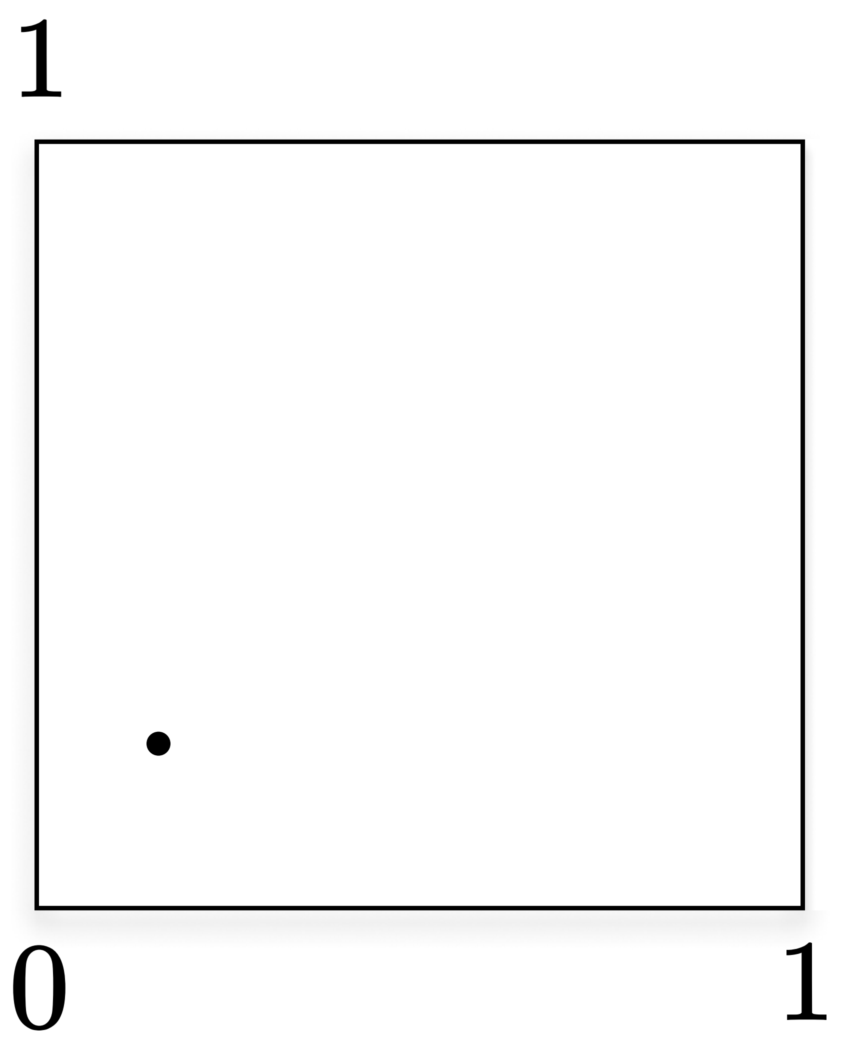
$$x_i + \theta_i \mod 1$$



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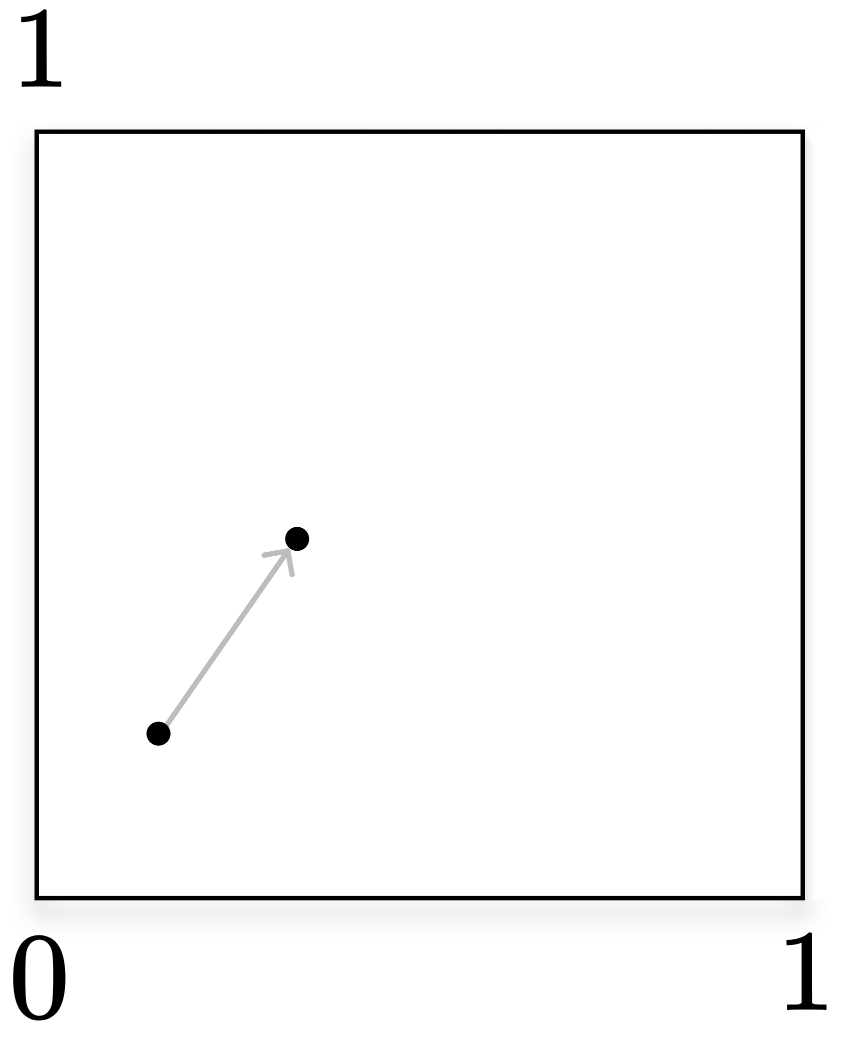

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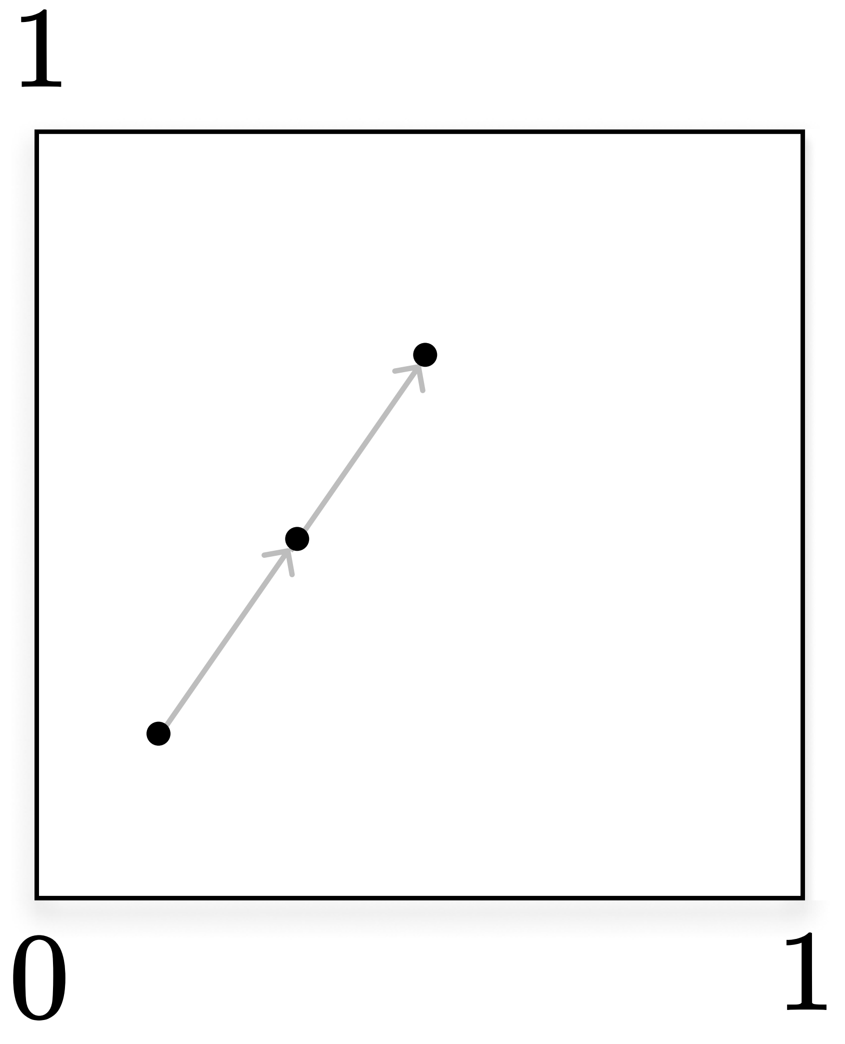

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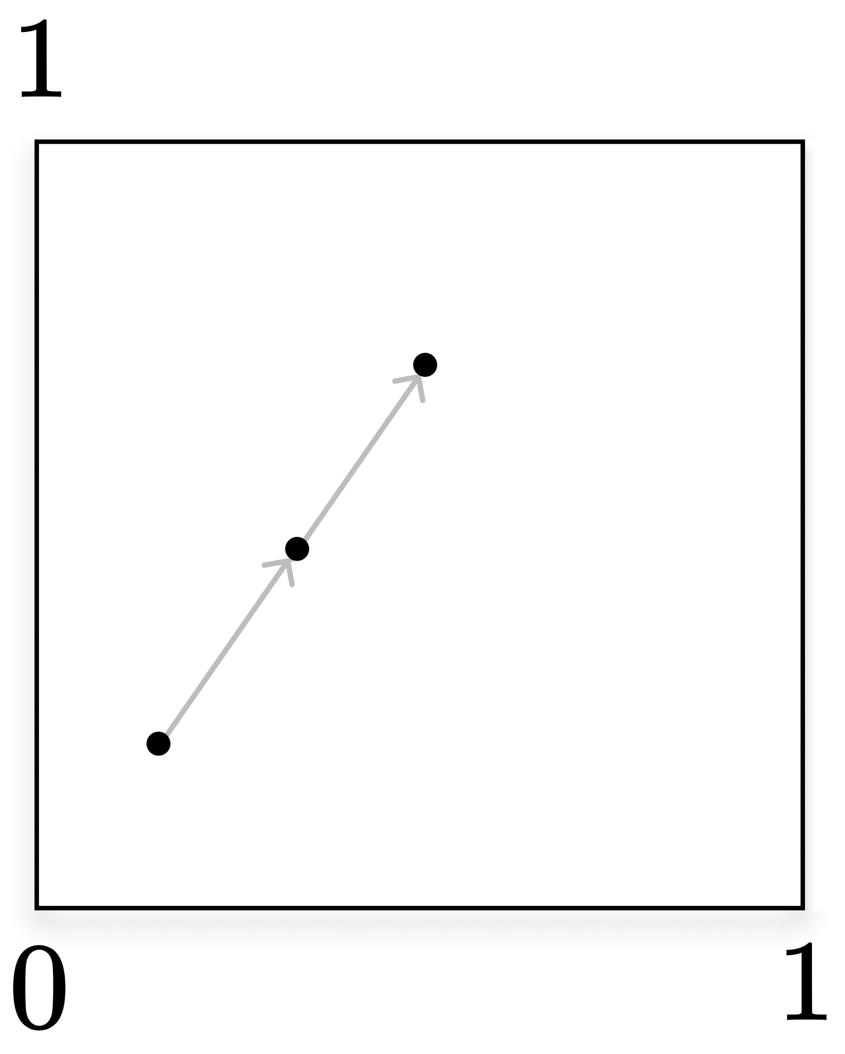

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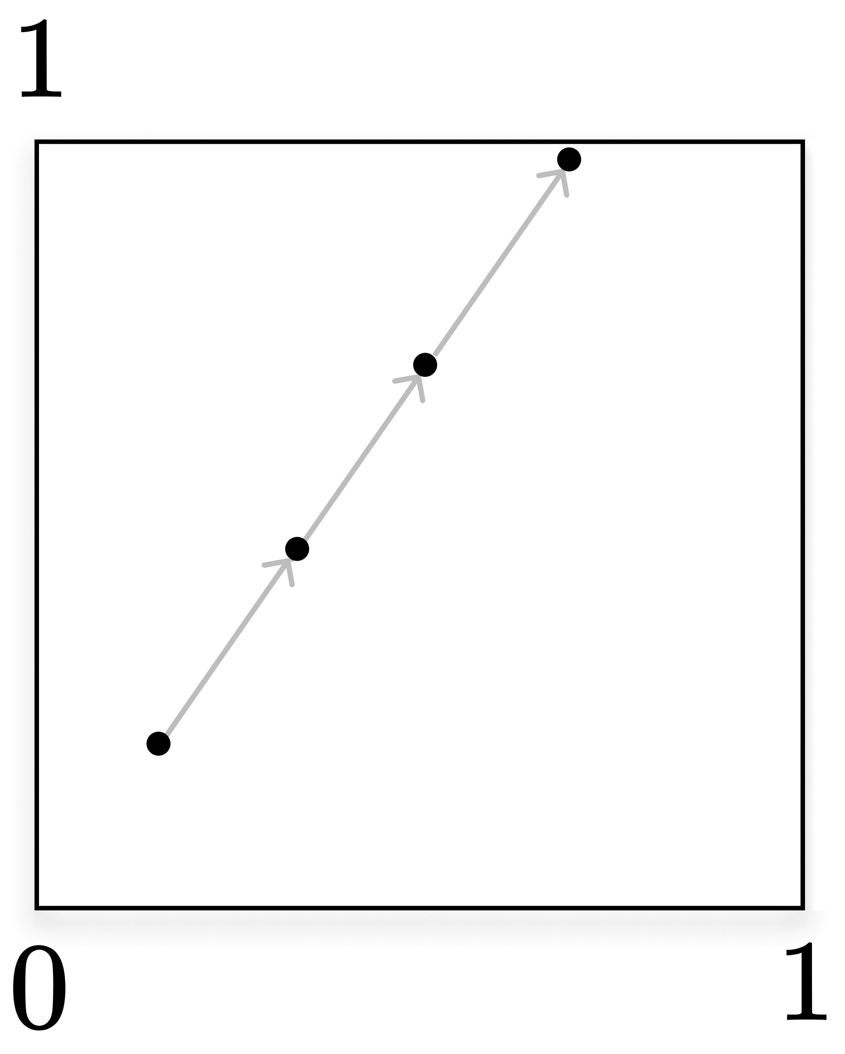

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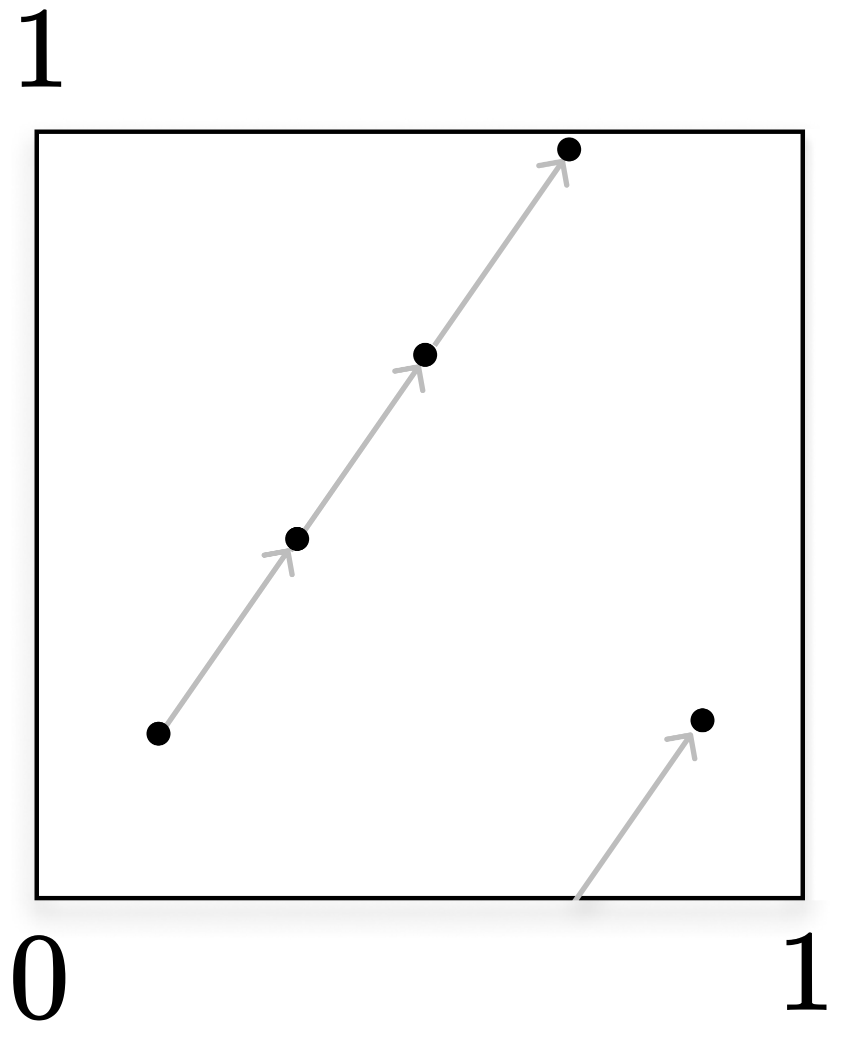

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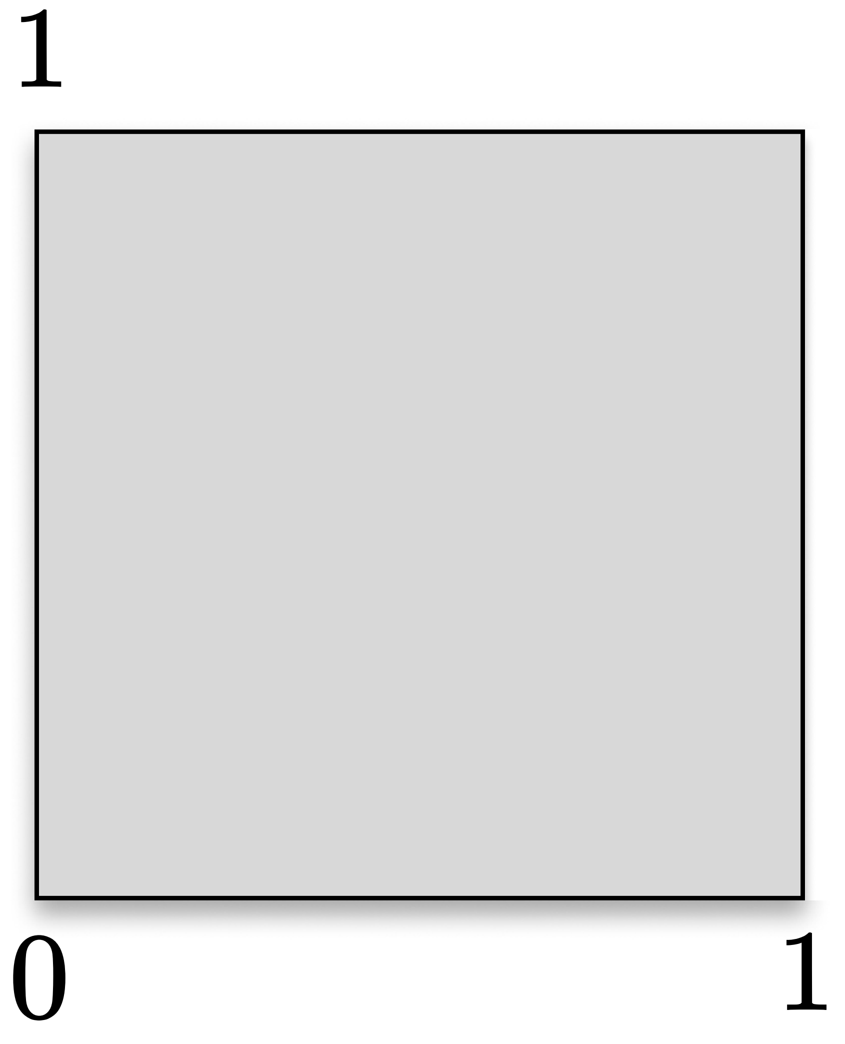

Theorem (Kronecker).
$(n\theta_1 \bmod 1, , \ldots, n\theta_k \bmod 1), \; n \in \mathbb{N}$
is dense in the hypercube

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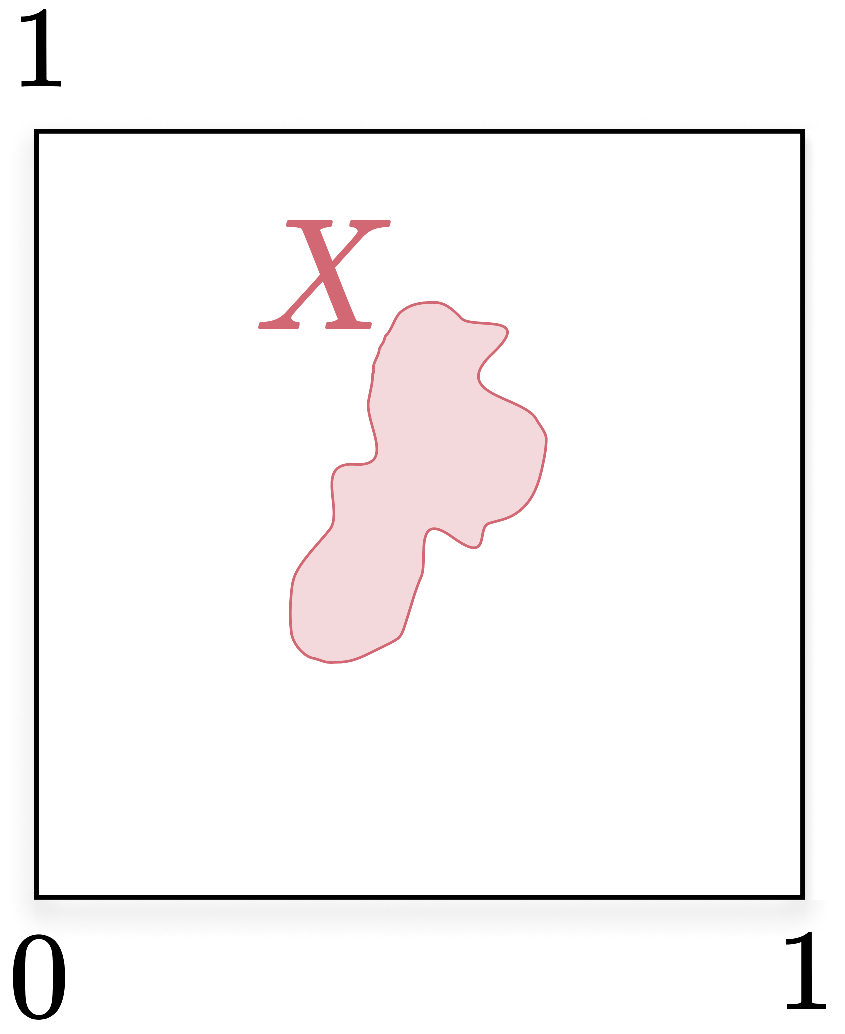

Theorem (Weyl, 1912).
$(n\theta_1 \text{ mod } 1, , \ldots, n\theta_k \text{ mod } 1), \; n \in \mathbb{N}$
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The amount of time spent in X is proportional to vol(X)

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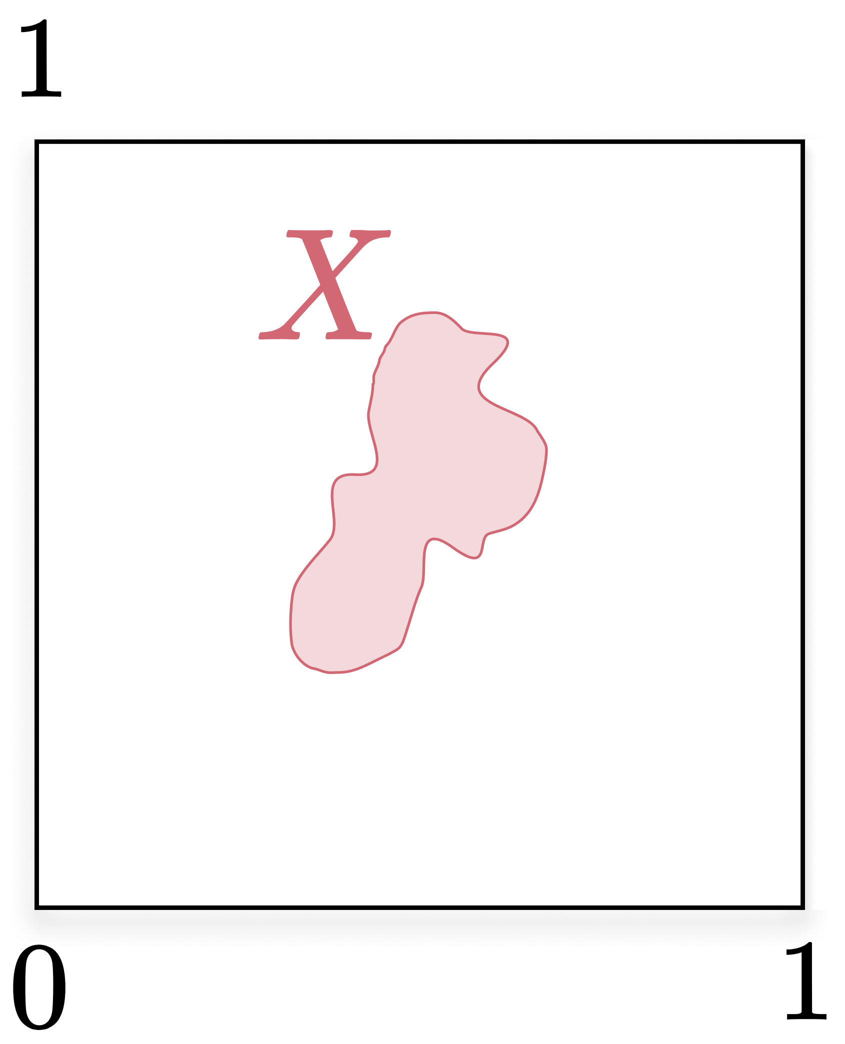

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Density of $\{n \; : \; (n\theta_1 \bmod 1, \ldots, n\theta_k \bmod 1) \in X\} = \mathrm{vol}(X)$

$$\underbrace{\sum_{i\in I} c_i\alpha_i^n + \sum_{i\in D} c_i \prod_{j\in I} \alpha_j^{q_{i,j}} + c}_{D(n)} + R(n), \qquad n \in \mathbb{N}$$

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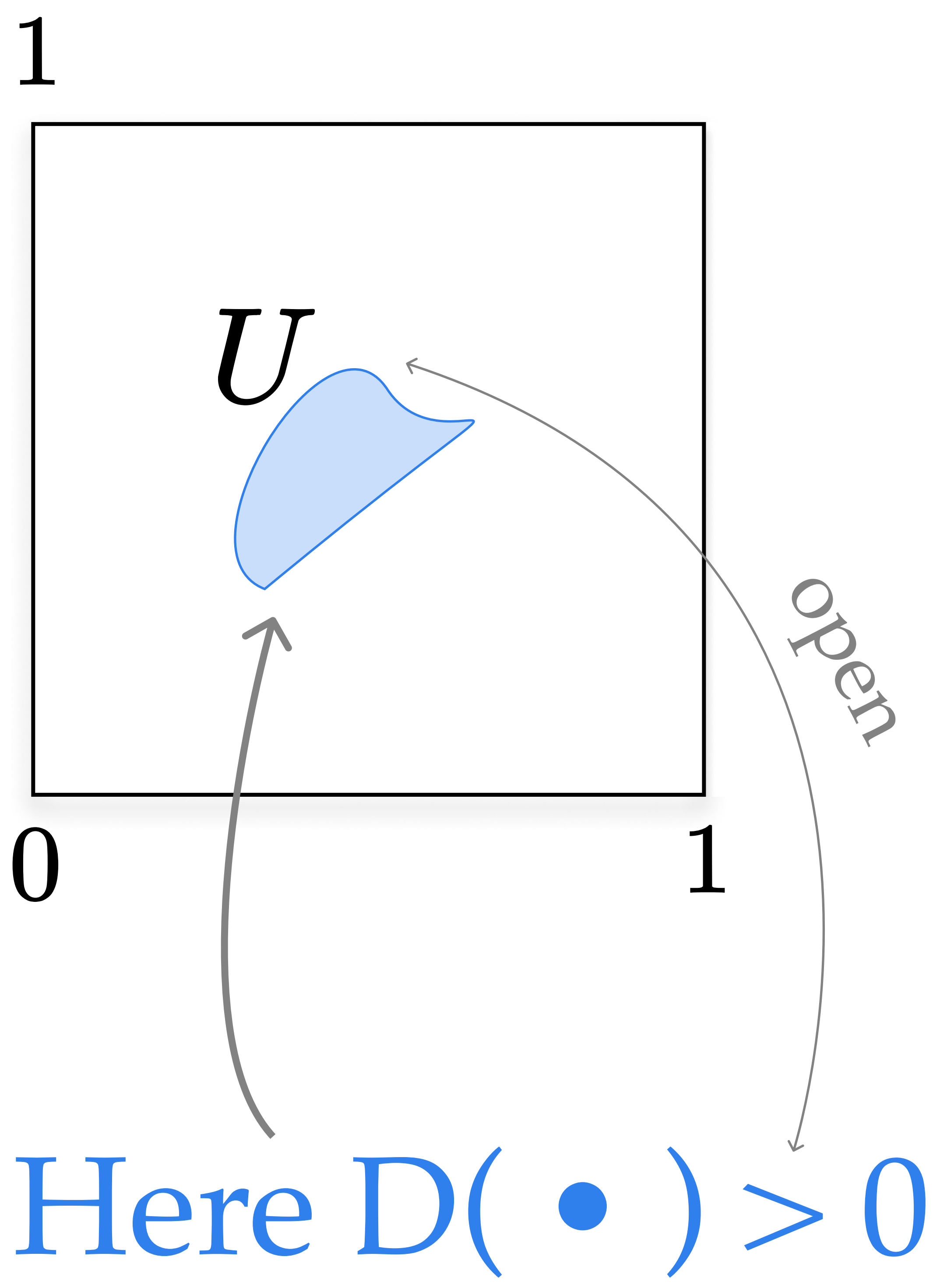
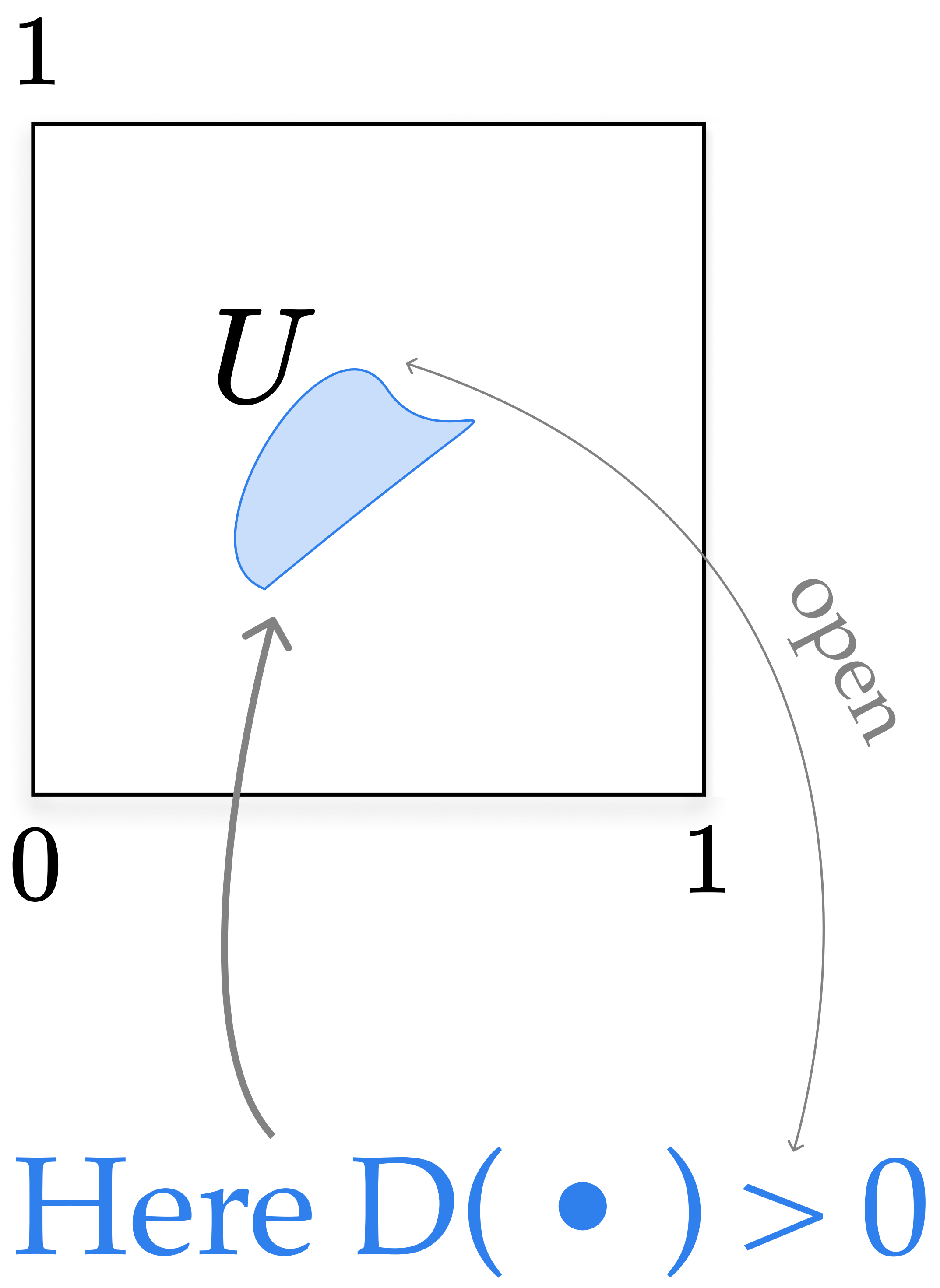


$$\text{Density of } \{n \;:\; D(n) > 0\} = \text{vol}(U)$$

$$\text{vol}(U) > 0 \;\Leftrightarrow\; \underbrace{U \neq \emptyset}$$

An equivalent statement can be decided with Tarski's algo

$$\underbrace{\sum_{i\in I} c_i \alpha_i^n + \sum_{i\in D} c_i \prod_{j\in I} \alpha_j^{q_{i,j}} + c}_{D(n)} + R(n), \qquad n \in \mathbb{N}$$



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$$\mathrm{vol}(U) > 0 \iff \underbrace{U \neq \emptyset}$$

An equivalent statement can be decided with FO of reals

so we can decide if the density of the positivity set of D(n) is nonzero

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We need: Density of $\{n \,:\, D(n) + R(n) > 0\}$

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Difficult problem: Depends on diophantine properties of α
(Main obstruction to decidability of Skolem, Positivity, etc.)

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- $\lim_{n \to \infty} |R(n)| = 0$ polynomially fast

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Theorem 1. " $\mathcal{D} = 0$? " is decidable.

(so is " $\mathcal{D} = 1$? " by symmetry)

Diagonalisable Matrices

$$\underbrace{\sum_{i\in I} c_i\alpha_i^n + \sum_{i\in D} c_i \prod_{j\in I} \alpha_j^{q_{i,j}} + c}_{D(n)} + R(n), \qquad n \in \mathbb{N}$$

When M is diagonalisable:

$\lim_{n\to\infty} |R(n)| = 0$ exponentially fast

In this case, the p-adic subspace theorem implies:

not effective

there is some N, such that for all n>N

$$|D(n)| > |R(n)| \,.$$

D(n)+R(n)>0 for infinitely many n

if and only if

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Theorem 1a does not hold for nondiagonalisable matrices.

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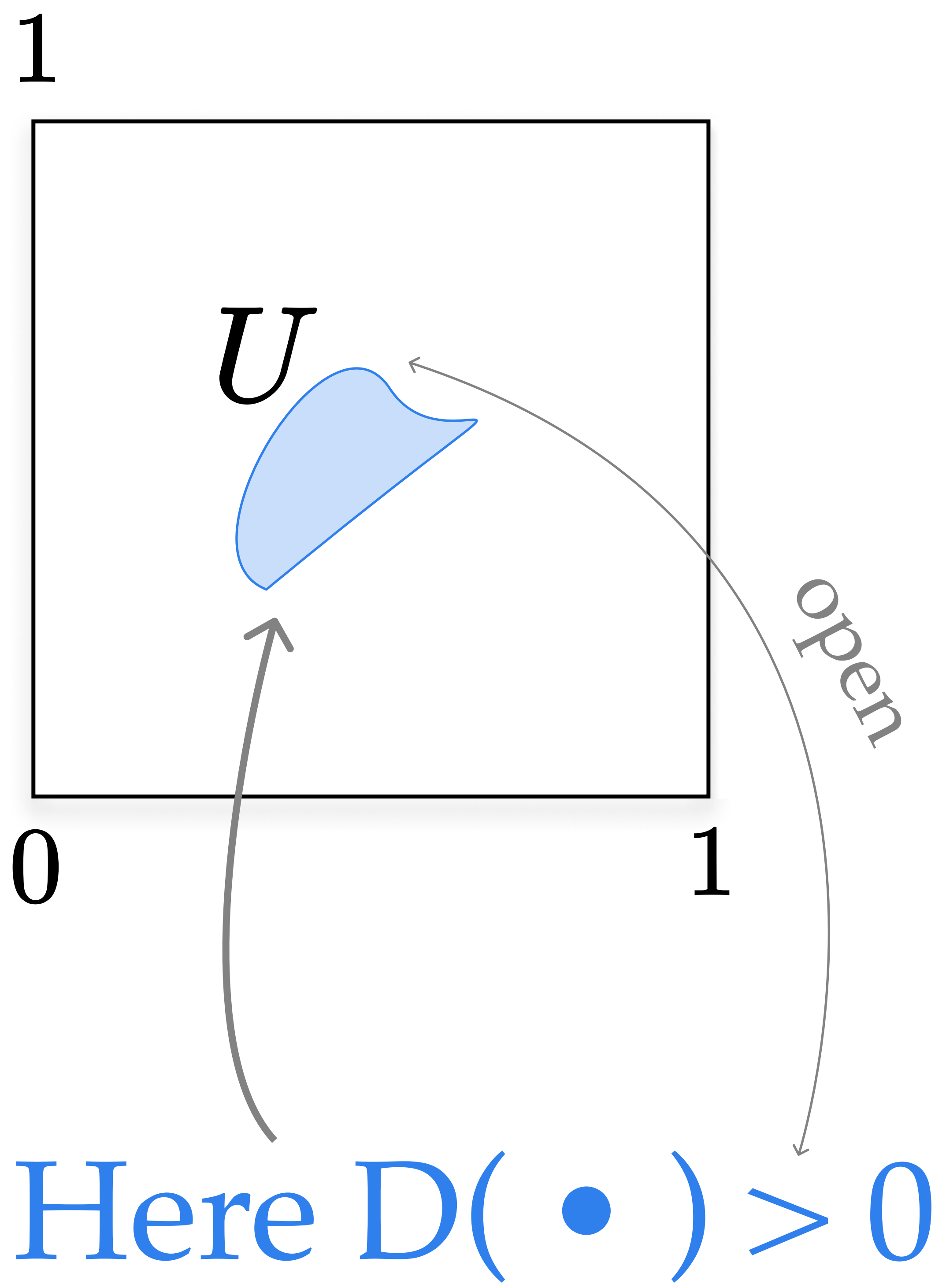
if and only if
D(n)>0 for infinitely many n

$\Rightarrow$

Joël Ouaknine, James Worrell:
Ultimate Positivity is Decidable for Simple Linear Recurrence Sequences. ICALP (2) 2014: 330-341

Theorem 1a. For diagonalisable update matrices:
$\mathcal{D} = 0 \Leftrightarrow$ finitely many ■

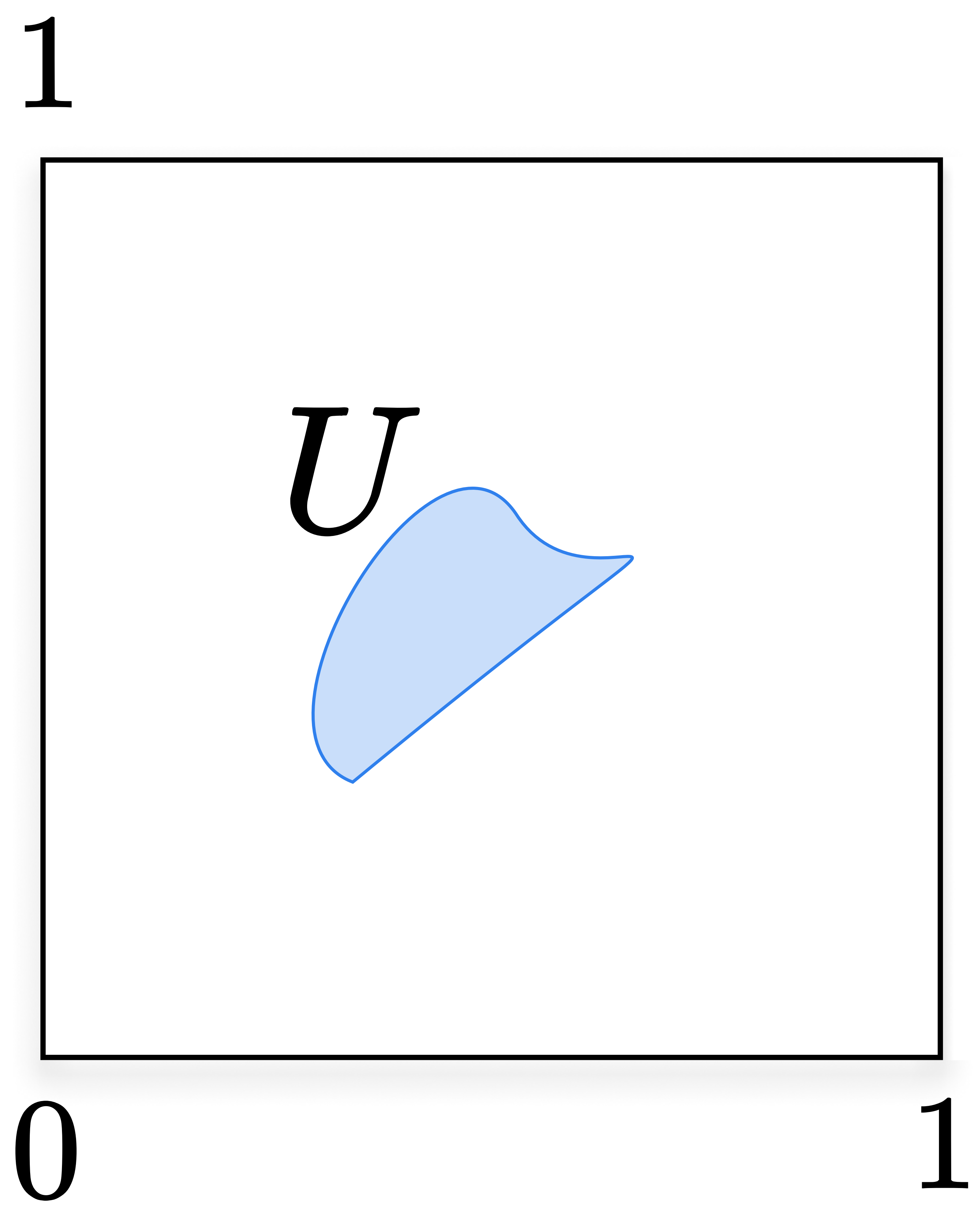
Approximating the Density



$$\overbrace{\text{Density of } \{n \;:\; D(n) > 0\} = \text{vol}(U)}^{\text{can be even transcendental}}$$

Approximating the Density

How to approximate the volume of U?



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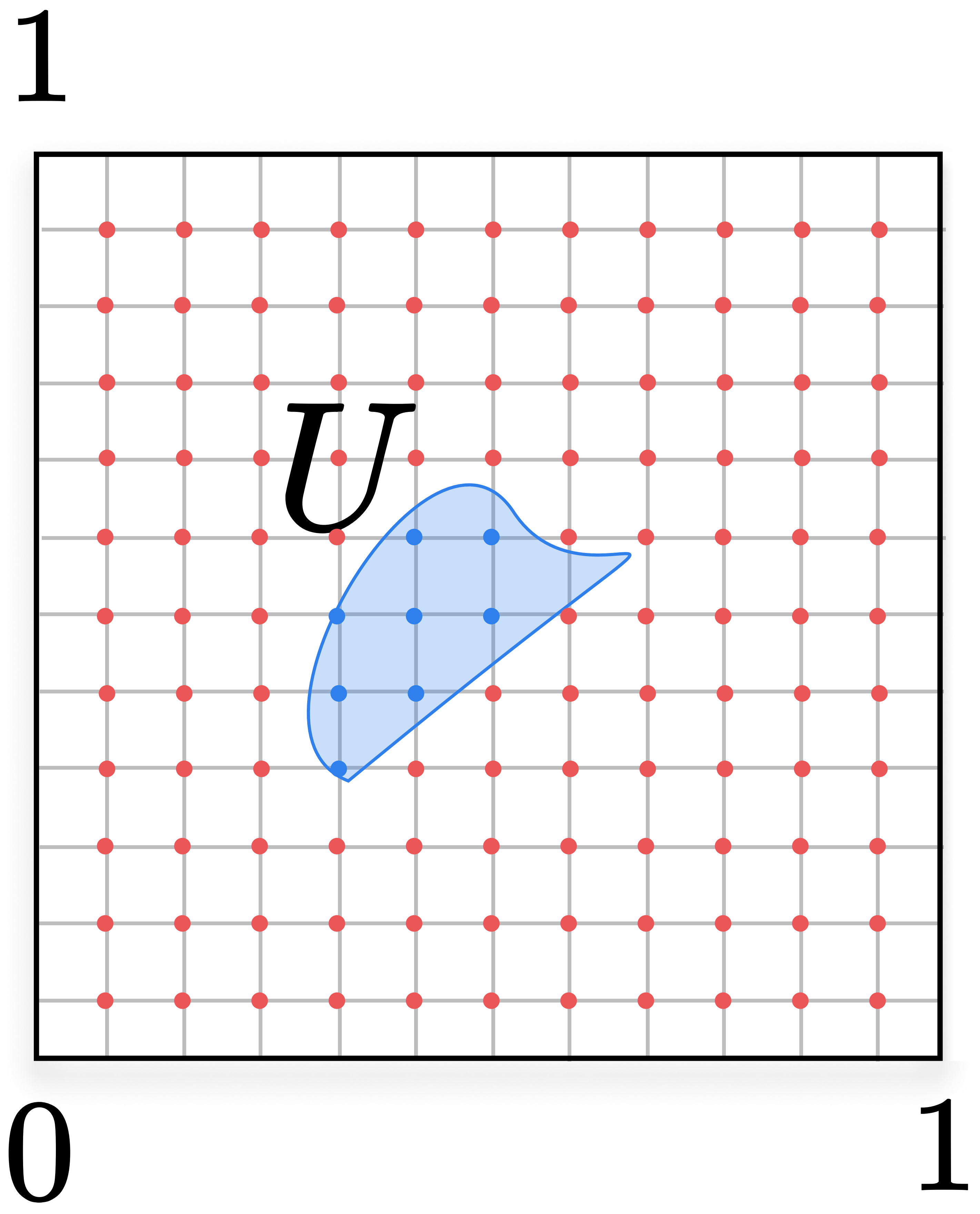


Approximate volume

=

number of ● (red) / number of ● (blue)

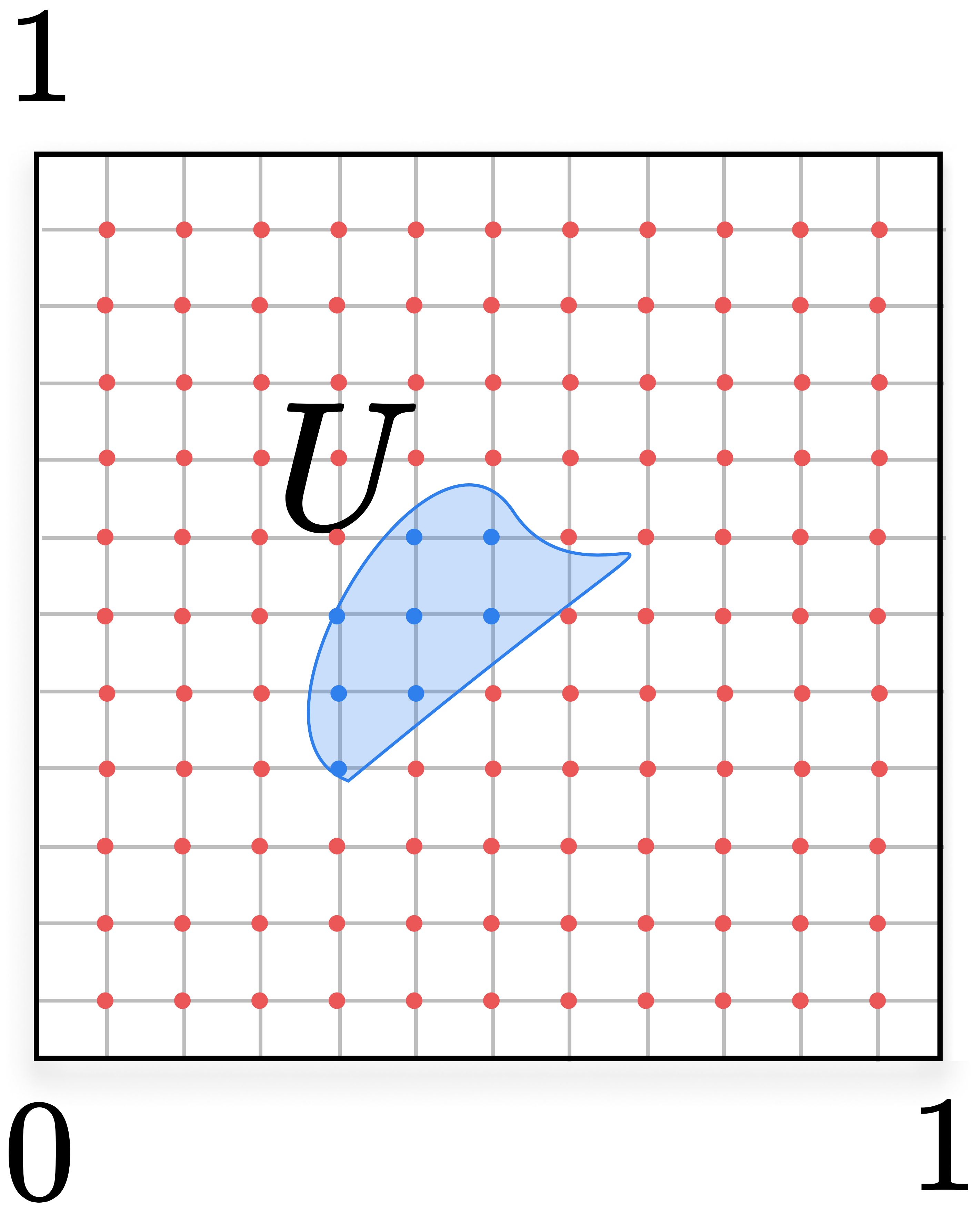
Approximating the Density

How to approximate the volume of U?

1. How to make a grid such that ● $\in U$ can be decided

2. How fine should the grid be for |approx - vol| < ε ?

Pascal Koiran:
Approximating the Volume of Definable Sets. FOCS 1995: 134-141



Approximate volume

||

$$\frac{\text{number of } \bullet}{\text{number of } \bullet}$$

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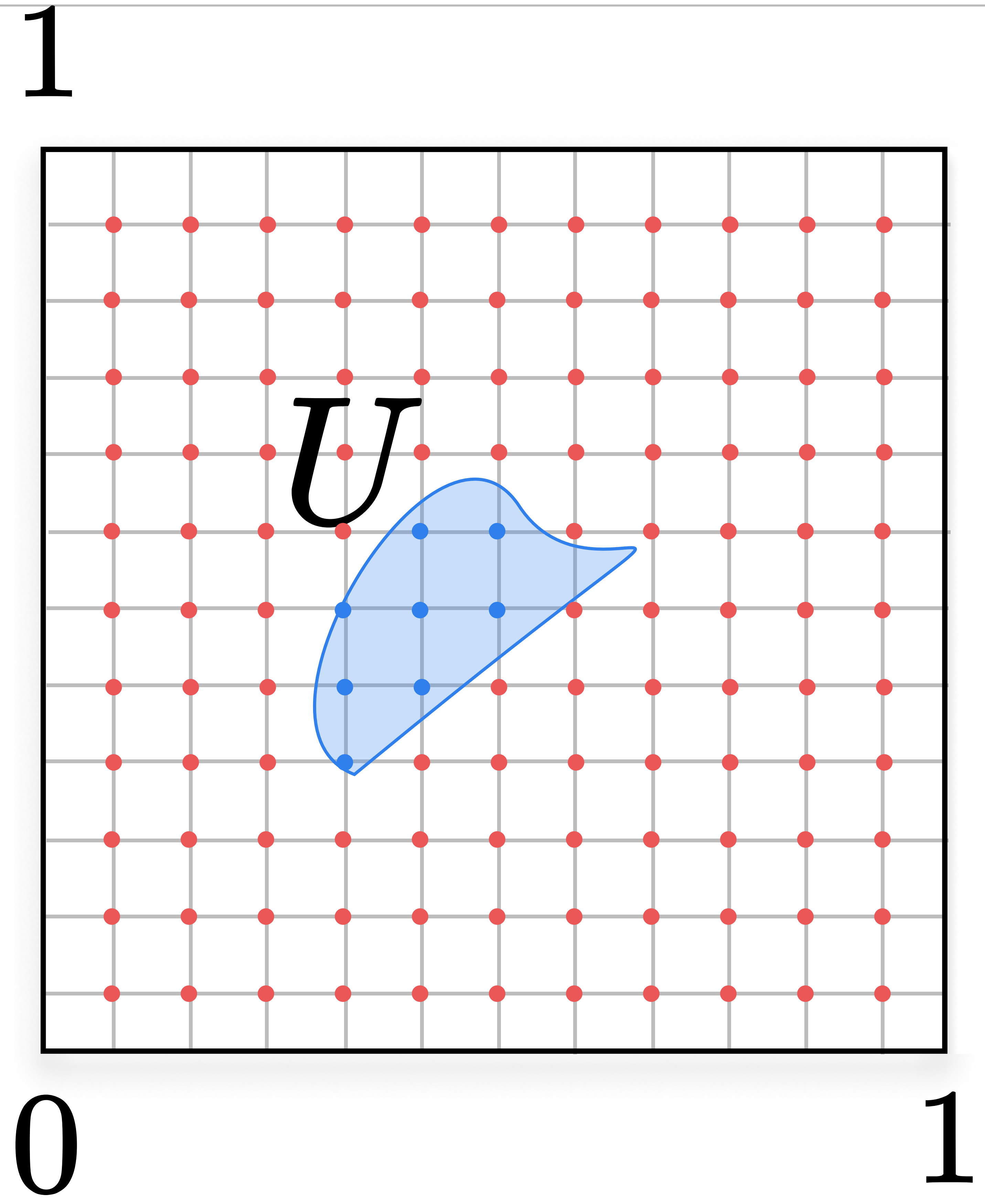
there is also a Monte-Carlo type algorithm



Approximate volume

=

$$\frac{\text{number of } \bullet}{\text{number of } \bullet}$$

Theorem 2. $\mathcal{D}$ can be computed to arbitrary precision.

Complexity

- < **PSPACE**
- when number of variables (order of LRS) is fixed, < **PTIME**
- > **co-NP**

V. The Open Problem

Decide whether

$$\mathcal{D} > \frac{1}{2}$$

Can be rational, algebraic, or transcendental

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Can be rational, algebraic, or transcendental

If $\mathcal{D} \notin \mathbb{Q}$ we can use the approximation algorithm

If $\mathcal{D} \in \mathbb{Q}$ then we can probably compute it directly

Can we decide whether $\mathcal{D} \in \mathbb{Q}$?

Decide whether $\mathcal{D} \in \mathbb{Q}$

When there are no multiplicative relations:

Decide if $$\int_{\mathcal{L}} \prod_{i=1}^{\eta} \frac{1}{\sqrt{1-x_i^2}} d\vec{x} \quad \in \quad \mathbb{Q}\,\pi^{\eta}$$

polytope

Decide whether $\mathcal{D} \in \mathbb{Q}$

When there are at most three dominant eigenvalues, the problem reduces to:

Given $\alpha \in \bar{\mathbb{Q}}$, decide whether $\cos^{-1}(\alpha) \in \mathbb{Q}\pi$

$\Updownarrow$

$T_n(\cos\theta) = \cos(n\theta)$

for some n, α is a root of

$T_n(x) - 1$ or $T_n(x) + 1$

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Theorem 3. "$\mathcal{D} \in \mathbb{Q}$?" is decidable, when there are at most three dominant eigenvalues.

Thank You

I. The Problem

```
x ← 0; y ← 6; z ← 4
while true do
    x ← 4x + 3y
    y ← 4y − 3x
    z ← 5z
    if y + z > 0 then
      Region A
    else
      Region B
    end if
end while
```

■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ . . .

Set of ■
1. Is it empty?
2. Is it infinite?
3. How big is it inside $\mathbb{N}$?

Decision questions:
1. Is Region A reached?
   (Is there at least one ■ ?)
   - Known as the **positivity problem**; at least as hard as Skolem's problem
2. Is Region A reached infinitely often?
   (Are there infinitely many ■ ?)
   - Known as the **ultimate positivity problem**; also open & difficult

In this paper:

3. How much more frequent are ■ compared to ■?

II. The Theorems

Theorems

Theorem 1. "$\mathcal{D} = 0$? " is decidable.
(so is "$\mathcal{D} = 1$? " by symmetry)

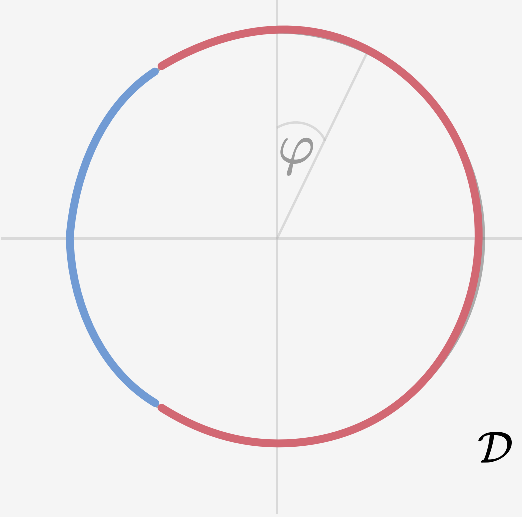
Theorem 1a. For diagonalisable update matrices:
$\mathcal{D} = 0 \Leftrightarrow$ finitely many ■

Theorem 2. $\mathcal{D}$ can be computed to arbitrary precision.

Theorem 3. "$\mathcal{D} \in \mathbb{Q}$?" is decidable,
when there are at most three dominant eigenvalues.

III. The Example, or First Observation

$\varphi = \cos^{-1} 4/5$



How frequently are we on the red arc?

Theorem (Weyl, 1910). Let ρ be an irrational real number. Then the sequence:
ρ, 2ρ, 3ρ, ...
is uniformly distributed mod 1.

$$\mathcal{D} = \frac{\text{length of }—}{2\pi} = \frac{\cos^{-1}(-2/3)}{\pi} = 0.732278\ldots$$

IV. The Proof

Approximating the Density

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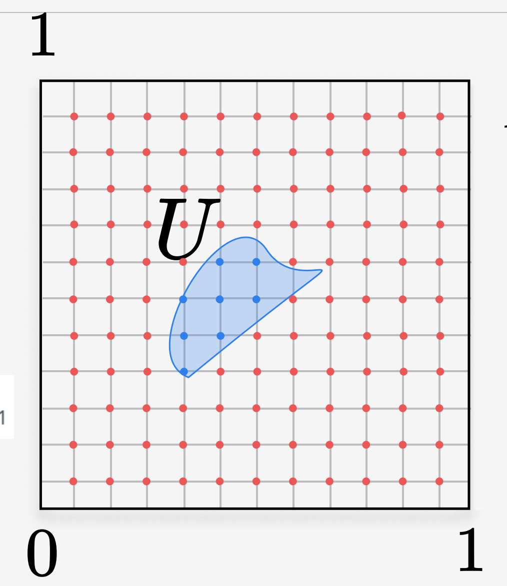

Approximate volume
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When there are no multiplicative relations:

Decide if $\displaystyle \int_{\mathcal{L}} \prod_{i=1}^{\eta} \frac{1}{\sqrt{1-x_i^2}} d\vec{x} \in \mathbb{Q}\pi^{\eta}$

polytope