MSO+∇ is undecidable

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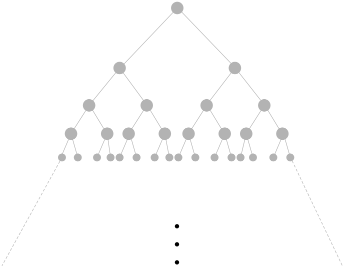
Monadic Second Order Logic on Trees



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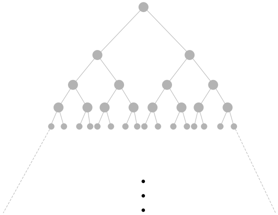

Quantify over *sets* of nodes

Quantify over nodes

$$\exists X \; \exists y \; \forall x \; (y > x) \Rightarrow (x \in X)$$

y is a descendant of x

x belongs to X

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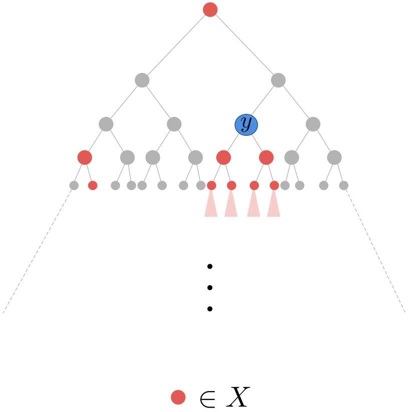

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Rabin's theorem

Theorem (Rabin 1969)

The problem:

input: *An MSO formula ϕ*
output: *Is ϕ true in the full binary tree*

is decidable.

$\Rightarrow$ decidability of LTL, CTL^*, modal μ-calculus, ...

Question

Is there a *probabilistic* extension of Rabin's theorem that subsumes probabilistic logics?

Measure quantifier

- Henryk Michalewski and Matteo Mio. Measure quantifier in monadic second order logic, LFCS, 2016.

$$\forall X \ \Phi(X) \quad \equiv \quad \Phi(X) \text{ holds for all sets of nodes } X$$

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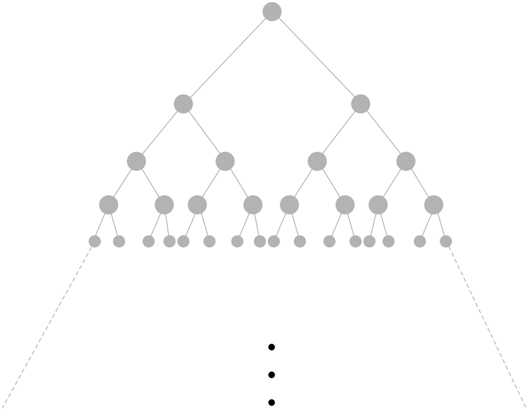
a new quantifier $\equiv$ $\Phi(X)$ holds almost surely for a *randomly* chosen set of nodes X

An attempt



independent coin throw for every node

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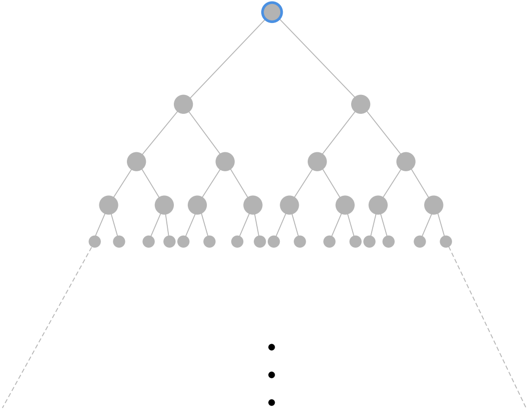

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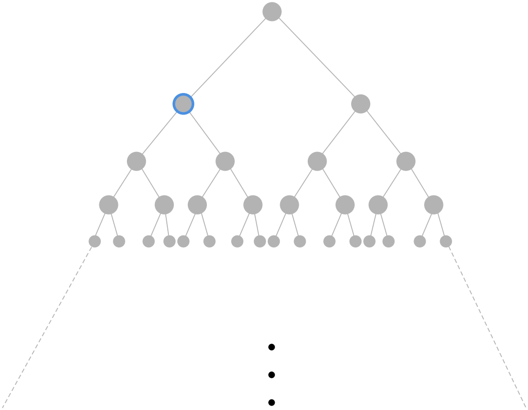

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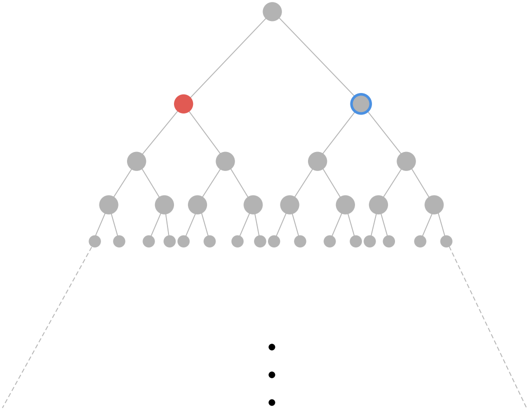

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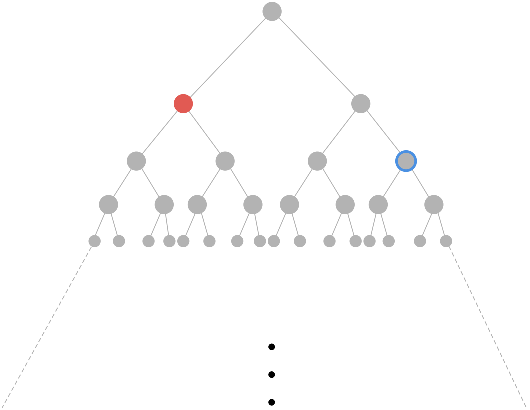

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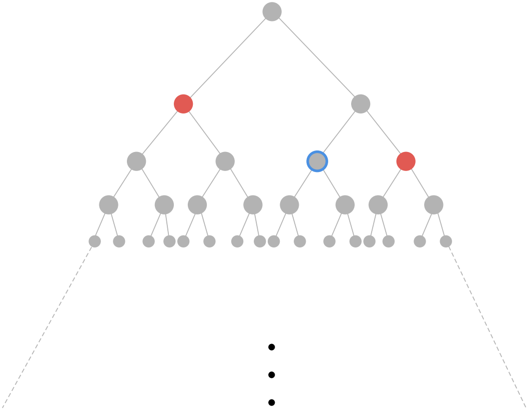

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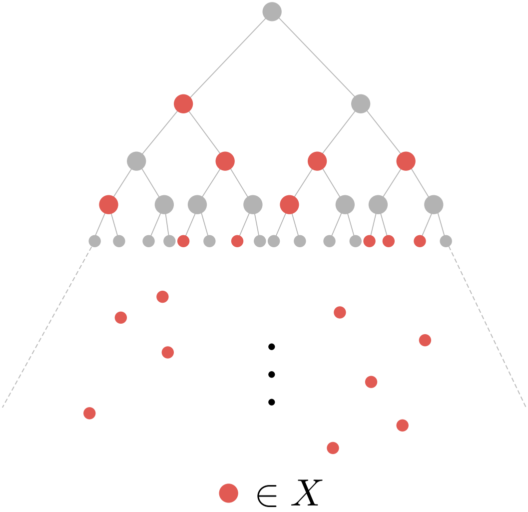

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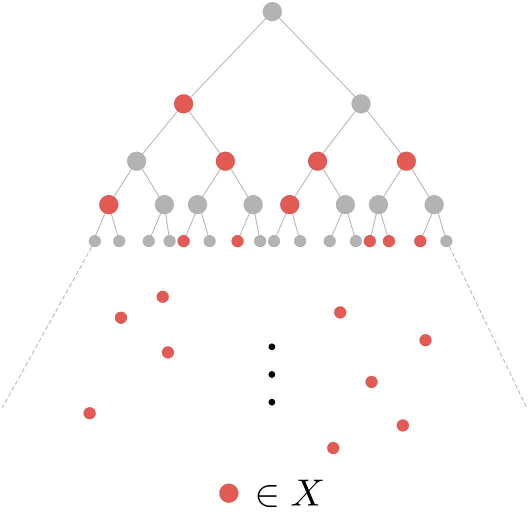

$\Rightarrow$ undecidable

A branch quantifier

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$\forall X\ \Phi(X) \equiv \Phi(X)$ holds for all X

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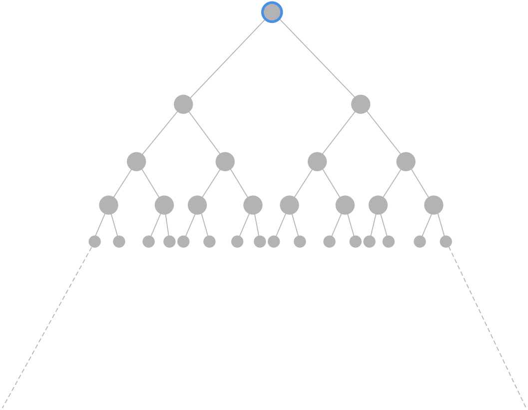

independent coin throw to choose a child at every step

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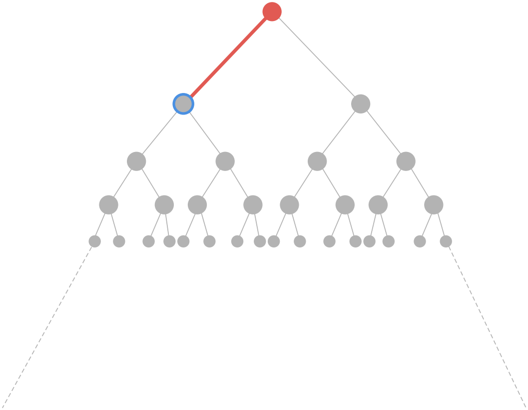

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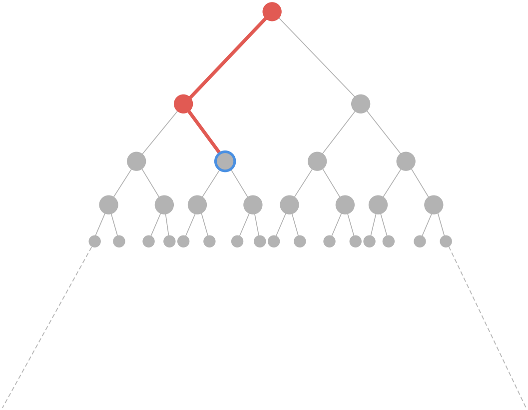

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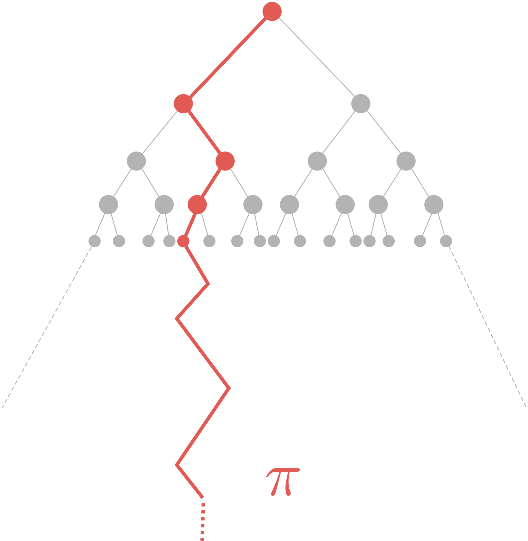
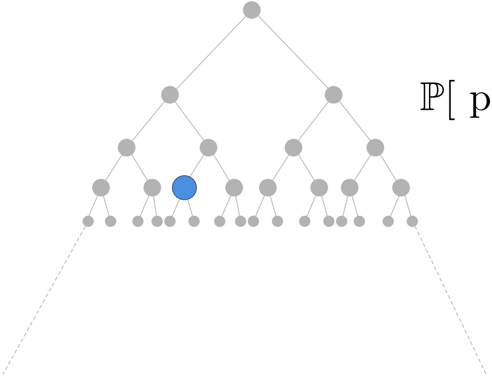


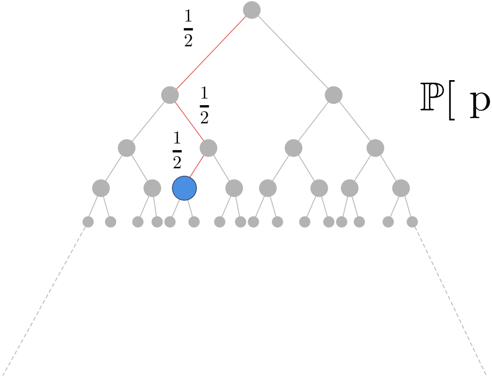
independent coin throw to choose a child at every step

Example: probability measure



$$\mathbb{P}[\text{ pass through } \bullet \ ] = \frac{1}{8}$$

Example: probability measure



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Definition: ∇ quantifier

$$\nabla \pi \ \Phi(\pi)$$

$$\equiv$$

$\Phi(\pi)$ holds almost surely for a randomly chosen branch π

Definition: ∇ quantifier

$$\nabla \pi \ \Phi(\pi)$$

$$\equiv$$

There exists a measurable set of branches Π such that

$\mathbb{P}[\Pi] = 1$ and every $\pi \in \Pi$ satisfies $\Phi(\pi)$

Example

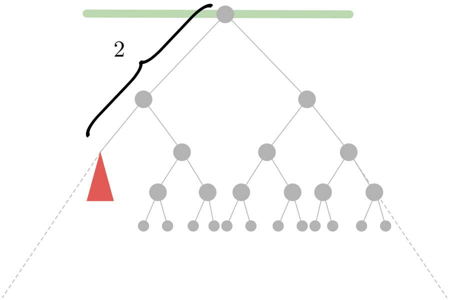
every node has a descendant in X

$$\exists X \begin{cases} \forall x \; \exists y \;\; (y \geq x \;\wedge\; y \in X) \\ \neg \nabla \pi \; (\exists x \;\; x \in \pi \;\wedge\; x \in X) \end{cases}$$

with nonzero probability X is avoided

Example: formula holds

$\exists X$ { every node has a descendant in X; with nonzero probability X is avoided }



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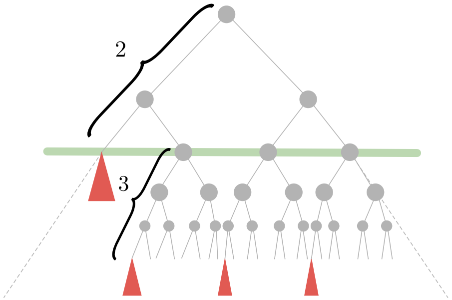

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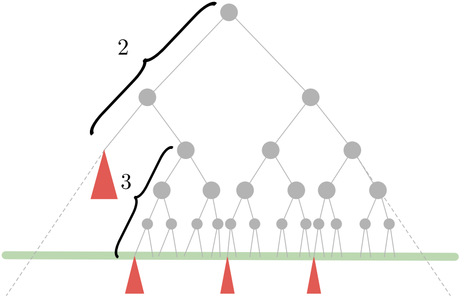

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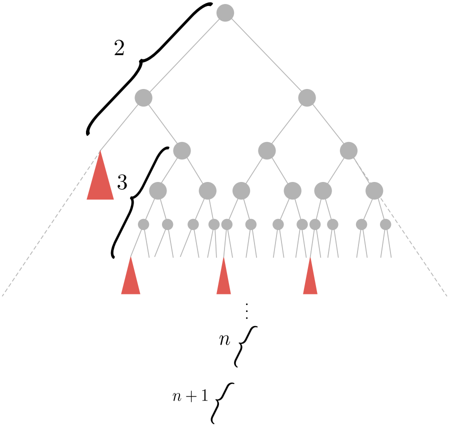

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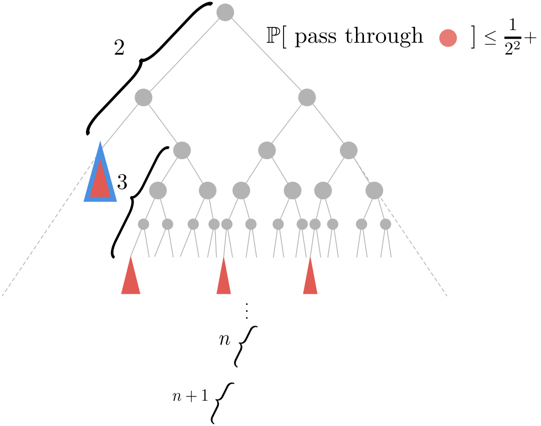

Example: formula holds

$\exists X$ { every node has a descendant in X
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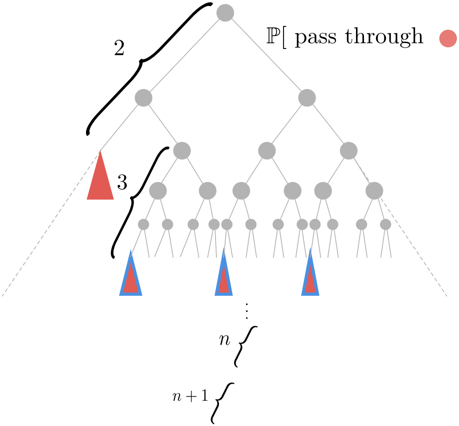
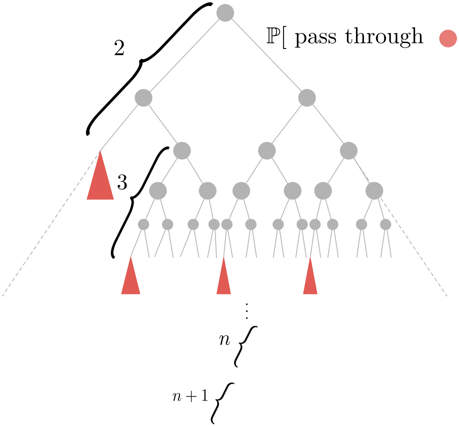
$\mathbb{P}[$ pass through ● $] \leq \frac{1}{2^2} + \frac{1}{2^3} +$



Example: formula holds

$\exists X$ {
- every node has a descendant in X
- with nonzero probability X is avoided



$\mathbb{P}[$ pass through ● $] \leq \sum_{n \geq 2} \frac{1}{2^n} = \frac{1}{2}$

Weak MSO+∇

$X, Y, Z, \ldots$ range over *finite* sets

Theorem (Bojańczyk 2016)

For every formula $\xrightarrow{\text{compute}}$ *equivalent suitable automaton*

Theorem (Bojańczyk, K, Gimbert 2017)

Emptiness of this automaton is decidable

Corollary

Weak MSO+∇ is decidable

Main theorem

Theorem

MSO+∇ is undecidable

Main theorem

Theorem

MSO+∇ is undecidable

- Independently and in parallel:

Raphaël Berthon, Emmanuel Filiot, Shibashis Guha, Bastien Maubert, Aniello Murano, Laureline Pinault, Jean-François Raskin, and Sasha Rubin. Monadic second-order logic with path-measure quantifier is undecidable.
https://arxiv.org/abs/1901.04349

A certain automaton has undecidable emptiness

Families of Intervals

Proof: MSO+∇ can express some asymptotic counting property.

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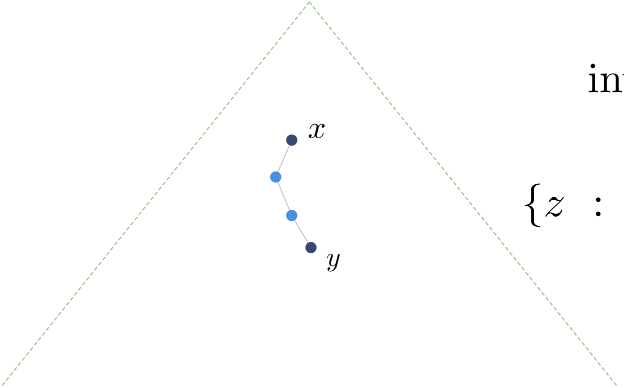

interval

$$\{z \ : \ x \leq z \leq y\}$$

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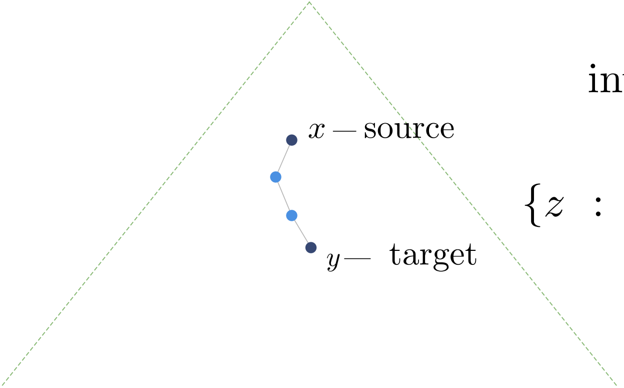

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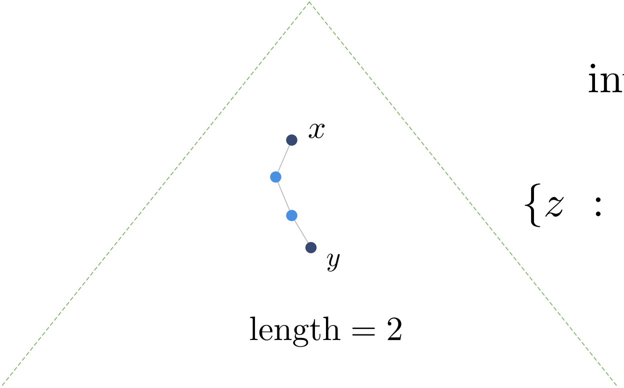

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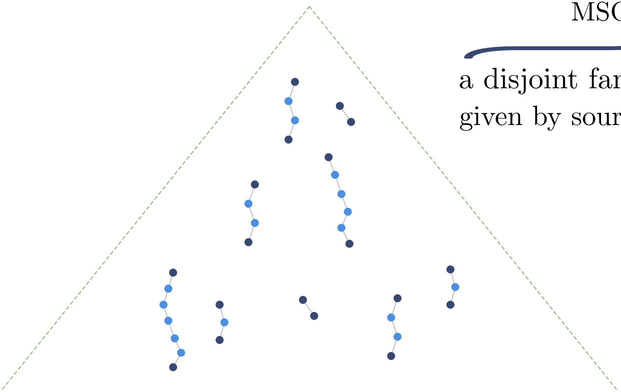

MSO-definable

a disjoint family of intervals $\mathcal{I}$
given by source($\mathcal{I}$) and target($\mathcal{I}$)

sets of nodes

Families of Intervals

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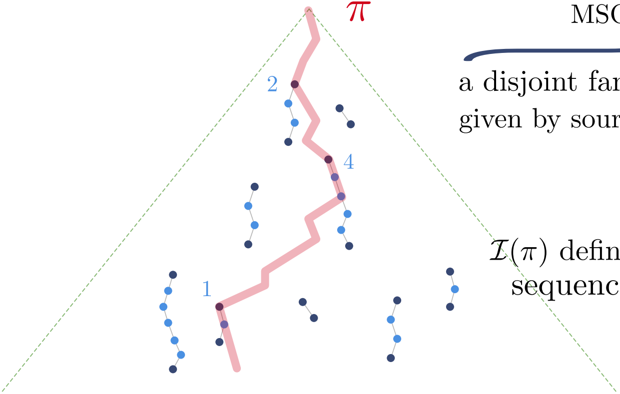

MSO-definable

a disjoint family of intervals $\mathcal{I}$ given by source($\mathcal{I}$) and target($\mathcal{I}$)

sets of nodes

$\mathcal{I}(\pi)$ defines an (in)finite sequence of numbers

$$\mathcal{I}(\pi) = 2, 4, 1, \ldots$$

$\mathcal{I}(\pi)$ is *eventually constant*:

$$\mathcal{I}(\pi) = 2, 4, 1, 7, \overbrace{5, 5, 5, \ldots}^{\text{only } 5}$$

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Theorem

There is a formula $\phi(X, Y)$ of MSO+∇ which is true if and only if

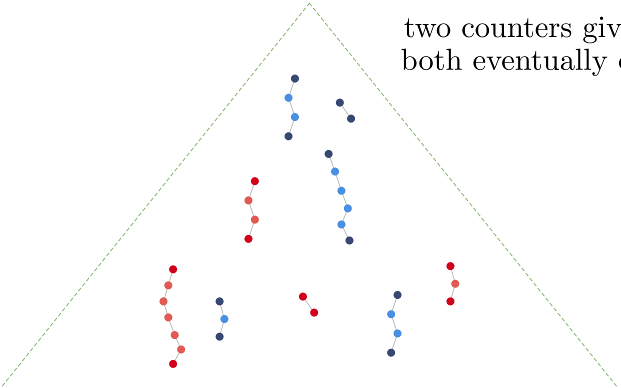
$$\mathbb{P}[\mathcal{I} \textit{ is eventually constant}] = 1$$

for some family of intervals $\mathcal{I}$ (that is unique if it exists) where

$$X = \mathrm{source}(\mathcal{I}) \qquad Y = \mathrm{target}(\mathcal{I}).$$

Counting

two counters given by $\mathcal{I}_1$ and $\mathcal{I}_2$
both eventually constant a.s



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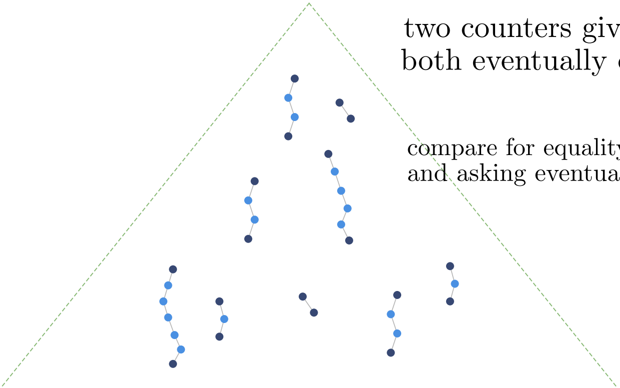

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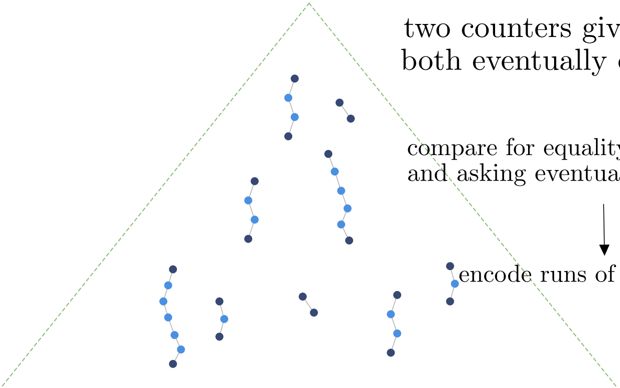

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↓

encode runs of a Minsky machine

Eventually constant property

$$\mathbb{P}[\mathcal{I} \text{ is eventually constant}] = 1$$

is asymptotic in two ways, it allows:

1. a set of branches with measure zero where the property does *not* hold
2. finite delay before the constant tail starts

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1. a set of branches with measure zero where the property does *not* hold
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we can count in a very weak way

Proof

express boundedness properties of $\mathcal{I}$

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express $\mathcal{I}$ is eventually constant

Proof

express boundedness properties of $\mathcal{I}$

use techniques from MSO+U

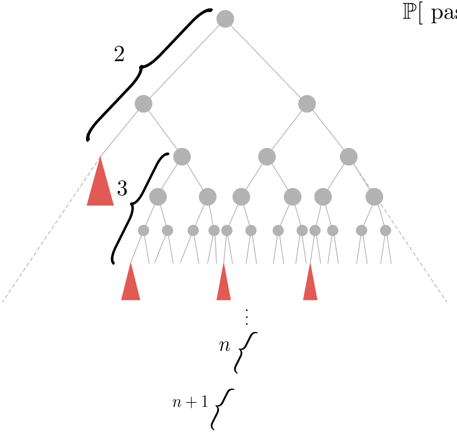
Mikołaj Bojańczyk, Paweł Parys, and Szymon Toruńczyk.
The MSO+U Theory of $(\mathbb{N}, <)$ Is Undecidable. STACS 2016

Mikołaj Bojańczyk, Laure Daviaud, Bruno Guillon, Vincent Penelle, and A. V. Sreejith
Undecidability of MSO+ultimately periodic, 2018

express $\mathcal{I}$ is eventually constant

Back to the Example

$\mathbb{P}[$ pass through ● $] \le \frac{1}{2}$



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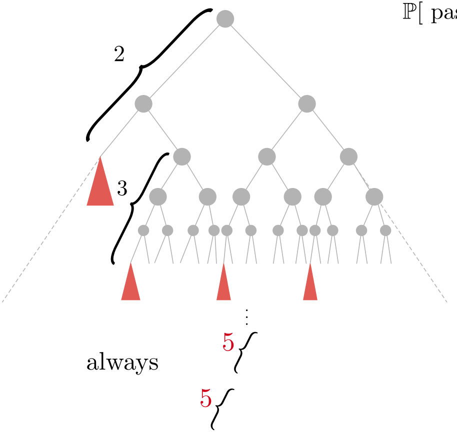

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Lemma

MSO+∇ can express

$$\mathbb{P}[\liminf \mathcal{I} < \infty] > 0.$$

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(∗) there exists $\mathcal{I}' \subseteq \mathcal{I}$ such that

$$\mathbb{P}[\ \underbrace{\mathcal{I}'\ \mathbf{io}}_{\substack{\text{a branch visits} \\ \text{sources of } \mathcal{I}' \\ \text{infinitely often}}}\ ] > 0$$

and all $\mathcal{K} \subseteq \mathcal{I}'$ satisfy

$$\mathbb{P}\big[\mathcal{K}\ \mathbf{io}\ \Rightarrow\ \mathrm{target}(\mathcal{K})\ \mathbf{io}\big] = 1.$$

Proof of Lemma

$$\mathbb{P}[\liminf \mathcal{I} < \infty] > 0 \Rightarrow \begin{cases} \exists\ \mathcal{I}' \subseteq \mathcal{I}\ .\ \mathbb{P}[\mathcal{I}'\ \mathbf{io}] > 0 \\ \forall\ \mathcal{K} \subseteq \mathcal{I}'\ .\ \mathbb{P}[\mathcal{K}\ \mathbf{io}\ \Rightarrow\ \mathrm{target}(\mathcal{K})\ \mathbf{io}] = 1 \end{cases}$$

By countable additivity of measures:

$$\exists n \in \mathbb{N} \quad \mathbb{P}[\liminf \mathcal{I} = n] > 0$$

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Say

$$\mathbb{P}[\liminf \mathcal{I} = 5] > 0$$

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Say

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Take $\mathcal{I}'$ to be intervals of length exactly 5

Proof of Lemma

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An interval is a record breaker if it is strictly longer than all of its ancestors

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Take $\mathcal{K}$ to be the record breakers of $\mathcal{I}'$.

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Take $\mathcal{K}$ to be the record breakers of $\mathcal{I}'$.

Proposition. $\mathbb{P}[\mathrm{target}(\mathcal{K})\ \mathbf{io}] = 0$. (as in example; $\mathcal{K}$ grows at least linearly)

Proof of Lemma

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Proposition. $\mathbb{P}[\mathrm{target}(\mathcal{K})\ \mathbf{io}] = 0$. (as in example; $\mathcal{K}$ grows at least linearly)

From the hypothesis: $\mathbb{P}[\mathcal{K}\ \mathbf{io}] = 0$ and $\mathbb{P}[\limsup \mathcal{I}' = \infty] = 0$.

Proof

express boundedness properties of $\mathcal{I}$ ✓

↓

express $\mathcal{I}$ is eventually constant

$$f = 2 \quad 7 \quad 9 \quad 10 \quad 15 \quad 0 \quad 4 \quad 18 \quad 29 \quad 105 \quad 20 \cdots$$

$$g = 10 \quad 24 \quad 42 \quad 13 \quad 7 \quad 1 \quad 0 \quad 0 \quad 2 \quad 5 \quad 5 \cdots$$

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$$f \sim g \quad \equiv \quad \forall \text{ sets of positions}$$

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$f \sim g \quad \equiv \quad \forall$ sets of positions

$7 \quad 10 \quad 15 \quad 29 \quad 20 \quad \cdots$ is bounded

if and only if

$24 \quad 13 \quad 7 \quad 2 \quad 5 \quad \cdots$ is bounded

$$f = 2 \quad 7 \quad 9 \quad 10 \quad 15 \quad 0 \quad 4 \quad 18 \quad 29 \quad 105 \quad 20 \cdots$$

$$g = 10 \quad 24 \quad 42 \quad 13 \quad 7 \quad 1 \quad 0 \quad 0 \quad 2 \quad 5 \quad 5 \cdots$$

$f \sim g \quad \equiv \quad \forall$ sets of positions

$7 \quad 10 \quad 15 \quad 29 \quad 20 \cdots$ is bounded

if and only if

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$$F = (2,3,4)\ (0,20,4)\ (1,1,4)\ (43,12,14)\ (2,19,17)\ (9,11,99) \cdots$$

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Mikołaj Bojańczyk, Paweł Parys, and Szymon Toruńczyk.
The MSO+U Theory of $(\mathbb{N}, <)$ Is Undecidable. STACS 2016

Lemma. For all n there is F of dimension n that is *not* an asymptotic mix of any G of dimension $< n$.

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boundedness

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boundedness *counting*

Encode $(3, 0, 2, 1)$ by

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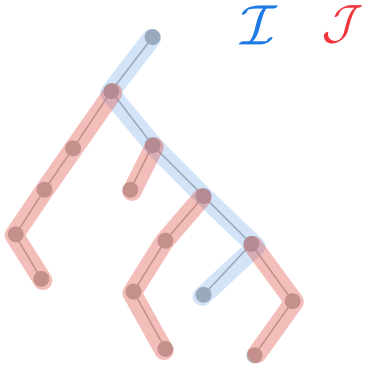

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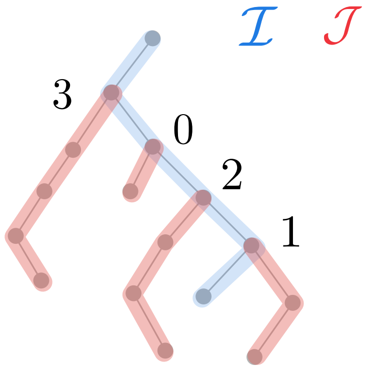

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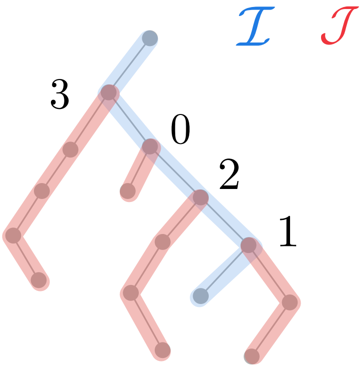

Length of $\mathcal{I} = \dim = 4$

Conclusion

Question

Is there *any* quantifier that can be added to MSO while retaining decidability?

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Is there *any* quantifier that can be added to MSO while retaining decidability?

- take some set of operations under which REG are closed
  - prove that any family of languages $\mathcal{F} \supset \mathrm{REG}$ closed under such operations must contain some undecidable language