The Density of Positive Entries of a Linear Recurrence

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$x \leftarrow 0;\ y \leftarrow 6;\ z \leftarrow 4$

while true **do**

$\quad x \leftarrow 4x + 3y$

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$\quad z \leftarrow 5z$

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$\quad$ **else**

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- arbitrary number of variables ranging over integers
- linear updates
- polynomial inequality

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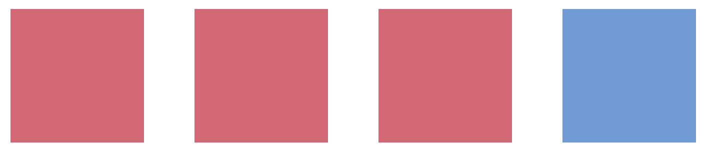

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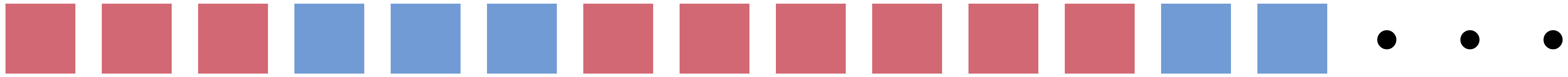

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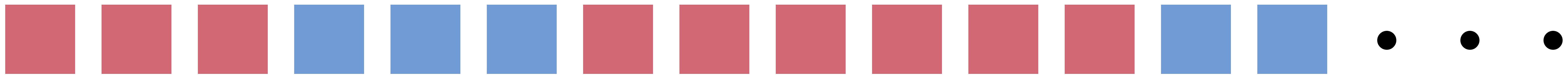

Decision questions:

1. Is Region A reached?
   (Is there at least one ?)

2. Is Region A reached infinitely often?
   (Are there infinitely many ?)

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In this paper:

3. How much more frequent are compared to ?

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Set of

1. Is it empty?
2. Is it infinite?
3. How dense is it?

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$$\frac{\text{Number of ■ in first } n \text{ entries}}{n}, \quad n \in \mathbb{N}$$

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$$\lim_{n\to\infty} \frac{\text{Number of } \blacksquare \text{ in first } n \text{ entries}}{n}$$

Exists due to:

Jason P Bell and Stefan Gerhold. On the positivity set of a linear recurrence sequence. *Israel Journal of Mathematics*, 157(1):333–345, 2007.

Called **density** denoted by $\mathcal{D}$.

Theorems

Theorem. $\mathcal{D} = 0?$ is decidable. (so is $\mathcal{D} = 1?$)

When the update matrix is diagonalisable: $\mathcal{D} = 0 \quad \Leftrightarrow \quad$ finitely many  .

Theorem. $\mathcal{D}$ can be computed to arbitrary additive precision.

Theorem. $\mathcal{D} \in \mathbb{Q}?$ is decidable,
when there are at most three dominant eigenvalues.

Applying the theorems to the example

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How frequently is Region A entered?

$$0.732279\cdots = \frac{\cos^{-1}(-2/3)}{\pi}$$

How does the algorithm work on the example?

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$$\underbrace{\begin{pmatrix} 0 & 6 & 4 \end{pmatrix} \cdot \begin{pmatrix} 4 & -3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}^n \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{\text{y+z in the nth loop iteration}}$$

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$$5^n \begin{pmatrix} 4/5 & -3/5 & 0 \\ 3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}^n = 5^n \begin{pmatrix} \cos n\varphi & \sin n\varphi & 0 \\ \sin n\varphi & \cos n\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation in the first two coordinates by $\varphi = \cos^{-1} 4/5$

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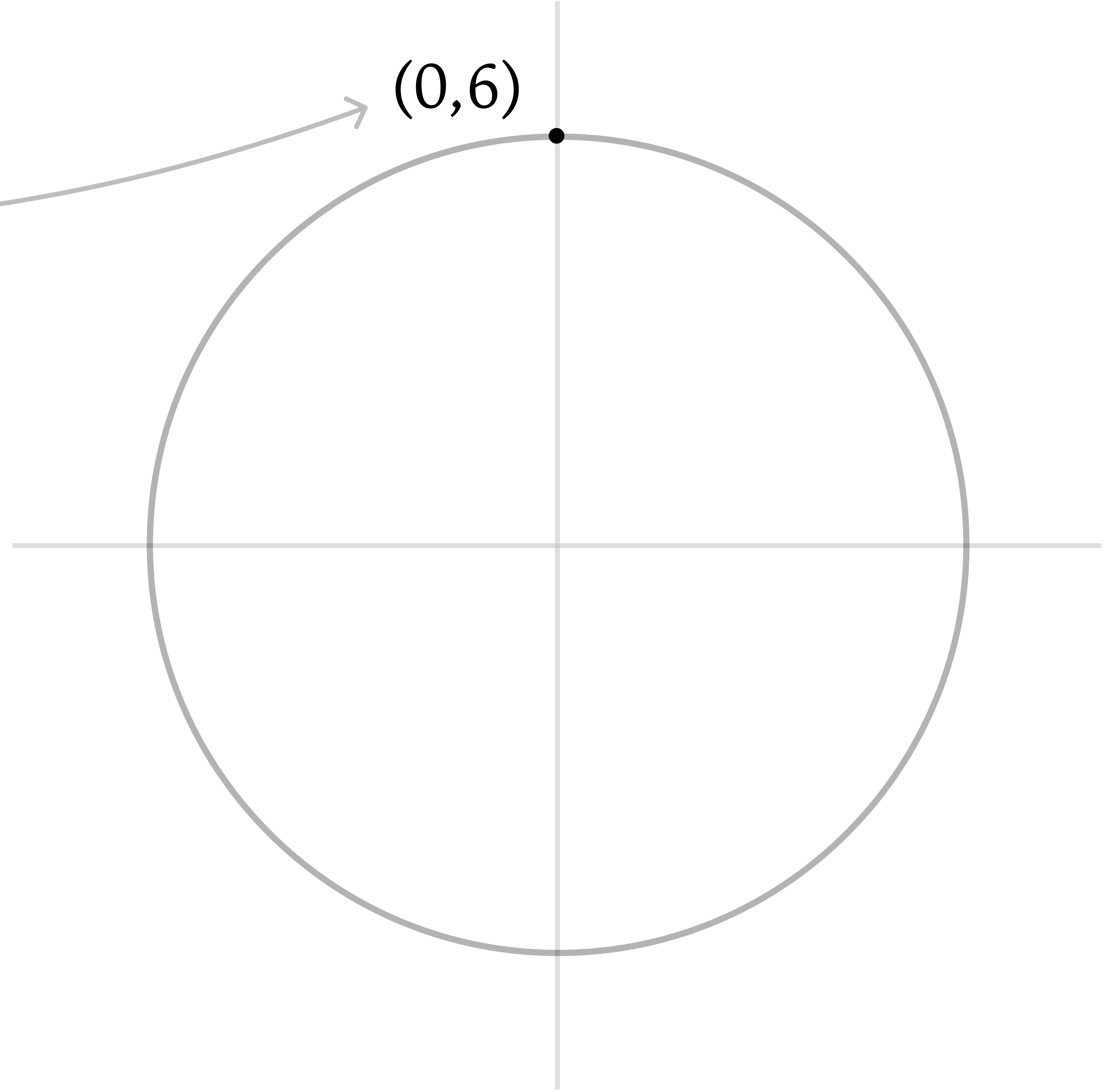

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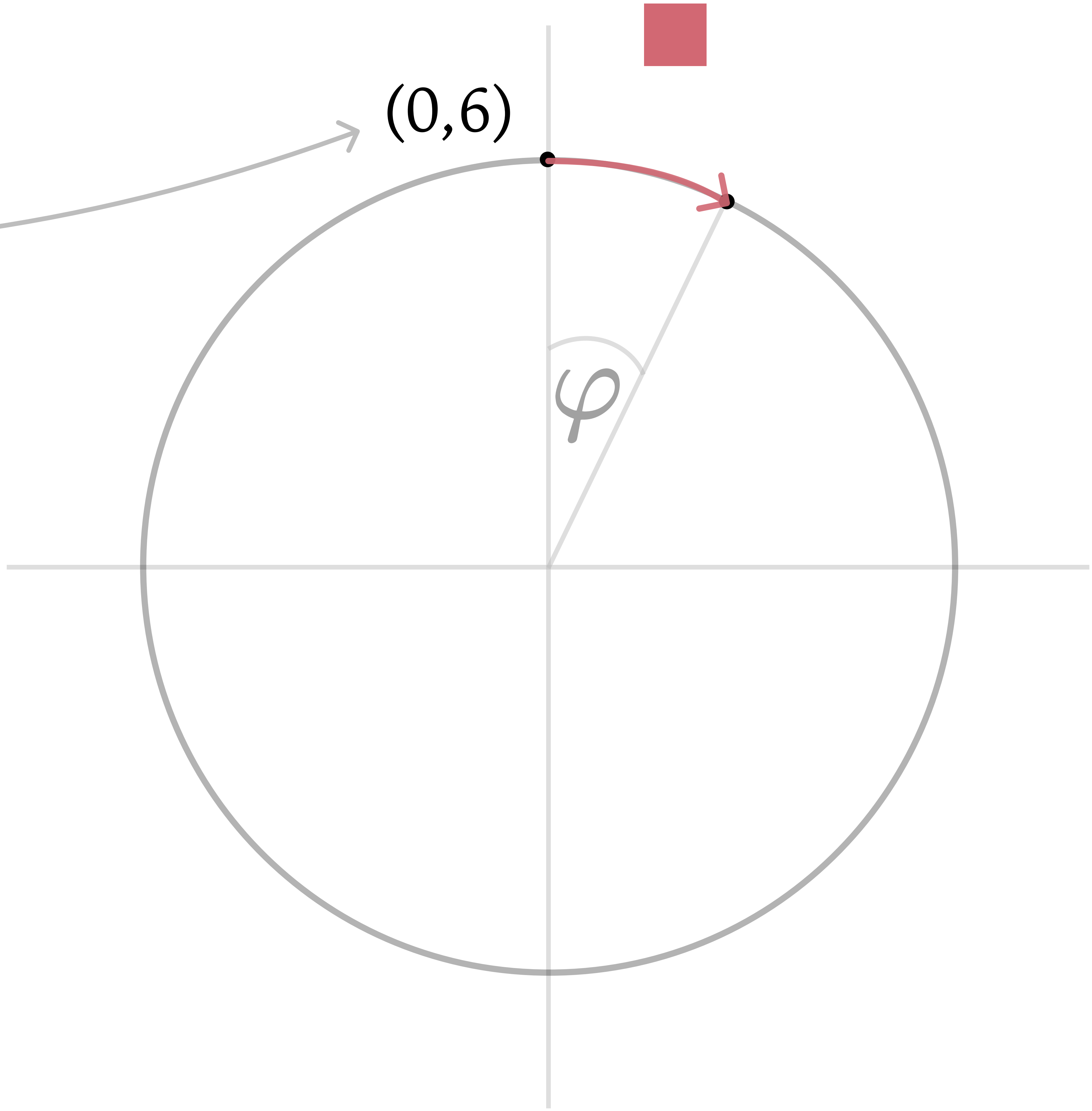

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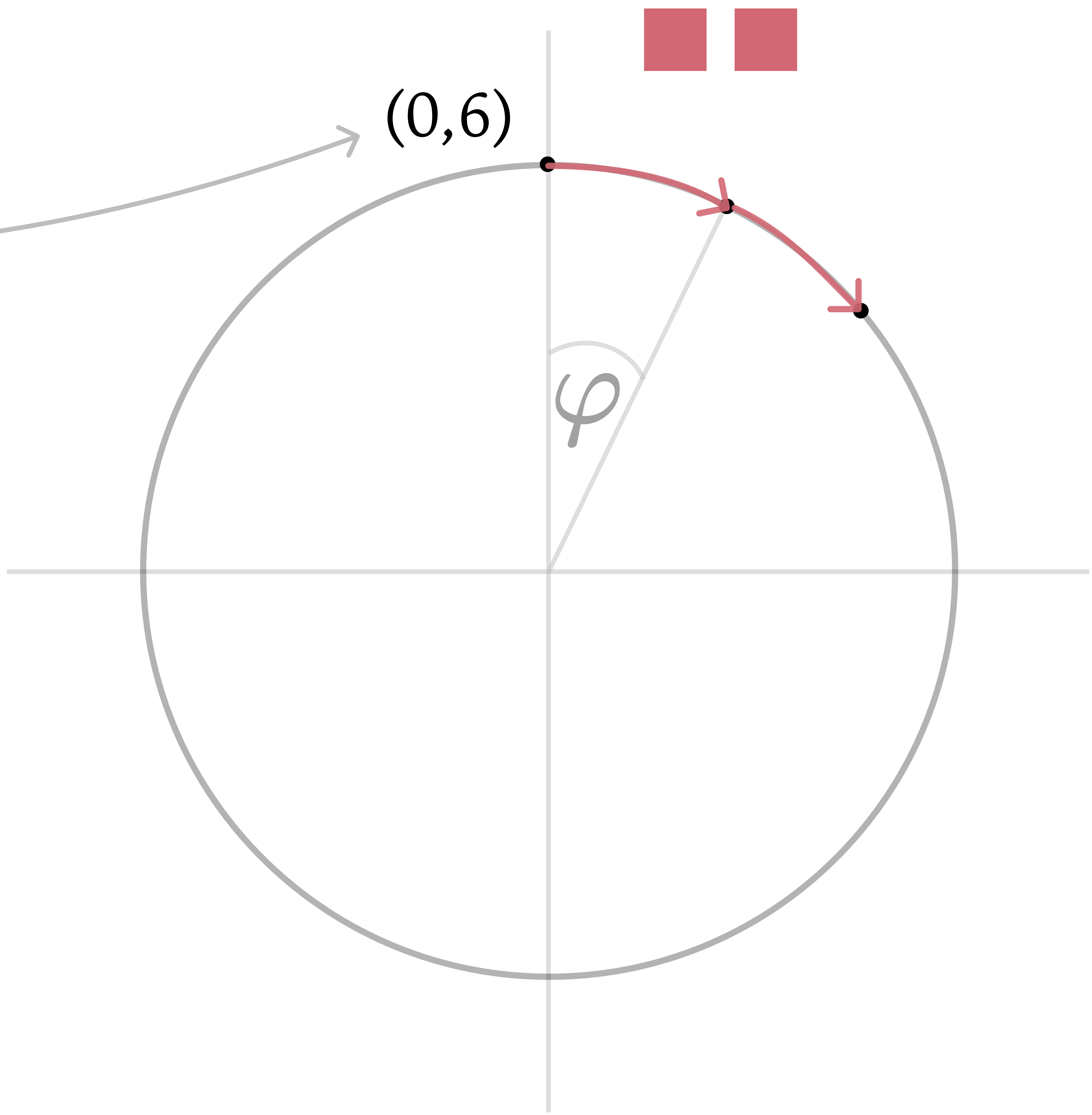

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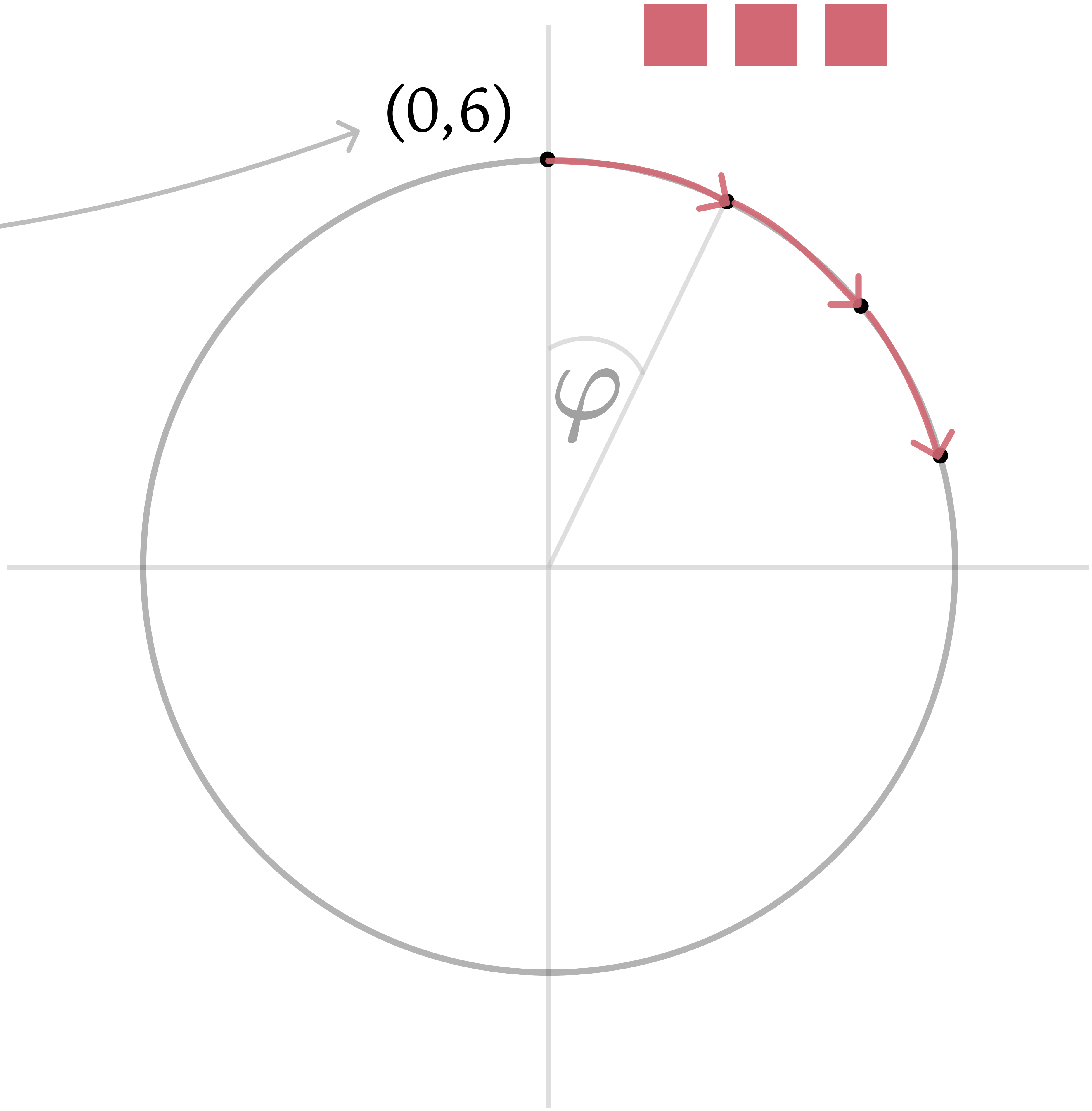

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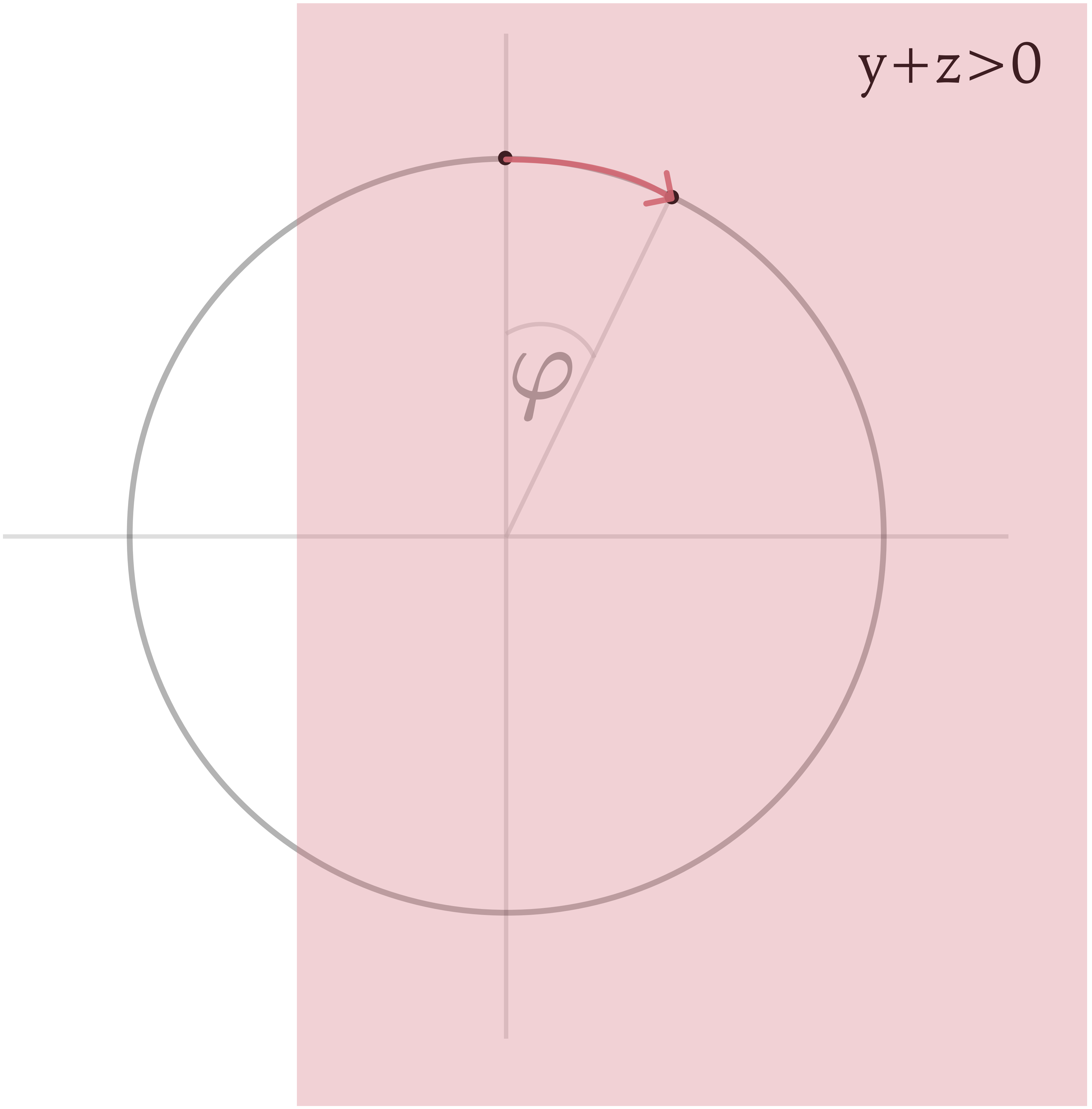

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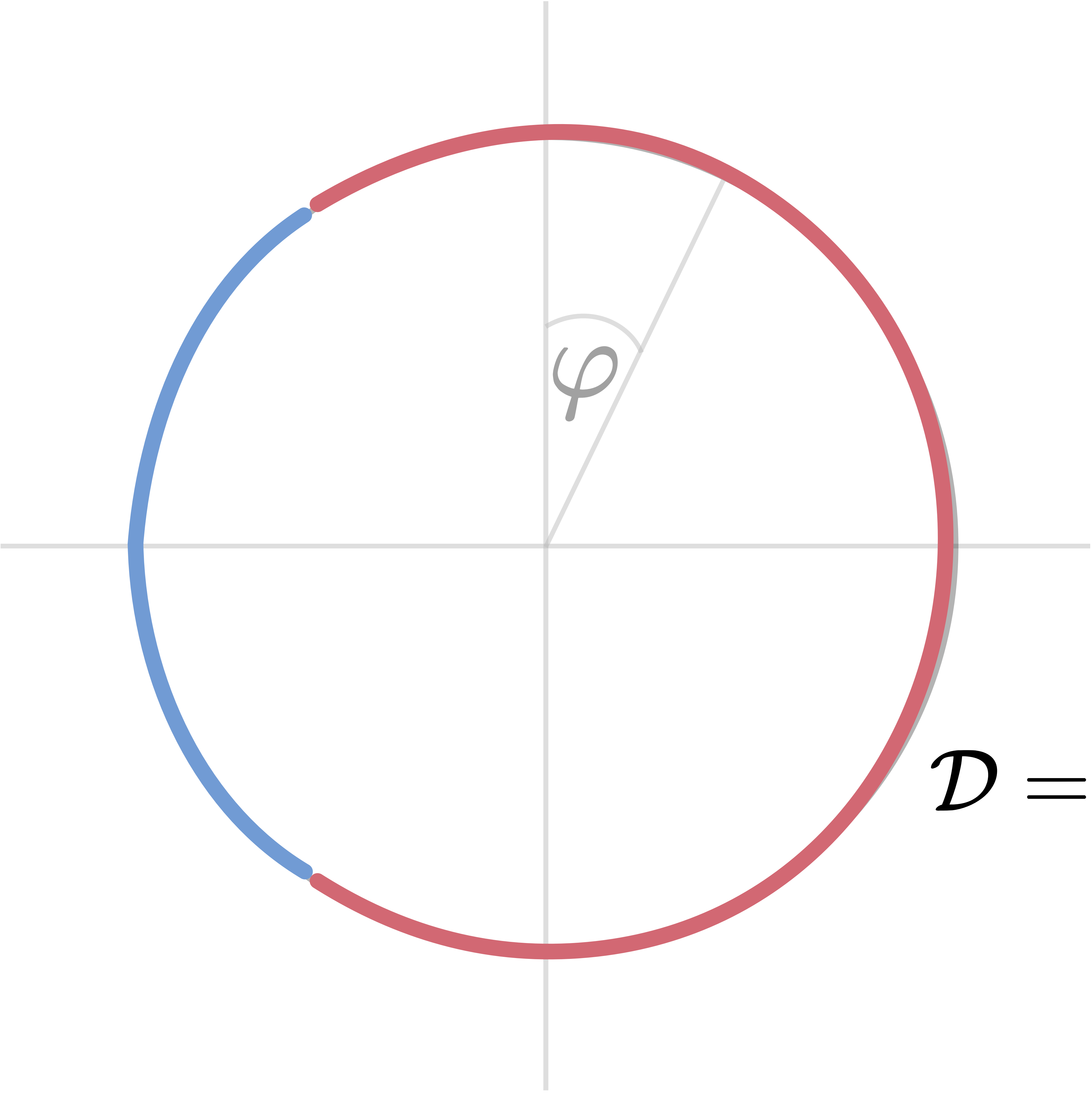
$\varphi = \cos^{-1} 4/5$



How frequently are we on the red arc?

Theorem. (Weyl) The frequency is proportional to the length of the arc.

$$\mathcal{D} = \frac{\text{length of }}{2\pi} = \frac{\cos^{-1}(-2/3)}{\pi} = 0.732278\ldots$$

For the general case we make crucial use of:

- A higher dimensional version of Weyl's theorem found in:

  J. W. S. Cassels. *An Introduction To Diophantine Approximation.* Cambridge University Press, 1959.

- Koiran's theorem

  Pascal Koiran. Approximating the volume of definable sets. In *Proceedings of IEEE 36th Annual Foundations of Computer Science*, pages 134–141. IEEE, 1995.

  to approximate the volume of certain constructible sets.

Open Problem

Can we decide whether $\mathcal{D} > 1/2$?

A priori can't use the approximation algorithm as $\mathcal{D}$ is not algebraic in general.

Theorem. The problem is solved in the case when there are at most three dominant eigenvalues by deciding whether:

$$\mathcal{D} \in \mathbb{Q}?$$

Thank you